Trapped surfaces in spacetimes with symmetries and applications to uniqueness theorems

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April, 2011

A mis padres.

A mi tía Nena.

“The transition is a keen one, I assure you,
from a schoolmaster to a sailor, and requires
a strong decoction of Seneca and the Stoics
to enable you to grin and bear it.
But even this wears off in time.”

Herman Melville, Moby Dick

Agradecimientos

Quiero empezar expresando mi más sincero agradecimiento al profesor Marc Mars Lloret, director de esta tesis doctoral, por la atención que me ha prestado durante todos estos años en los que he tenido la suerte de trabajar a su lado; por haber compartido conmigo sus ideas y haberme mostrado las líneas de investigación a seguir en este trabajo; por tratar siempre de animar mi curiosidad y mi carácter crítico por encima de todo; por su confianza, su paciencia y su apoyo, sin los cuales nunca hubiera podido terminar este trabajo; por su incansable dedicación, su total disponibilidad, su trato siempre amable y su sana amistad.

Quiero agradecer a todos los miembros del Departamento de Física Fundamental de la Universidad de Salamanca, por su trato cordial y por ayudarme siempre que lo he necesitado. Doy las gracias al profesor Walter Simon por su amistad y por compartir su saber conmigo. También agradezco al profesor Miguel Sánchez Caja del Departamento de Geometría y Topología de la Universidad de Granada, que fue mi tutor en los cursos de doctorado, por su hospitalidad, su apoyo y su sano interés por mi trabajo durante todos estos años.

Agradezco a mis compañeros de doctorado (Marsopas y otras especies): Cuchi, Jorge, Álvaro Dueñas, Álvaro Hernández, Diego, Alberto Soria, Edu y Cristina por su ayuda, por las risas y por aguantarme todos los días. A Toni, el Ave Fénix de las Marsopas, por haber compartido conmigo tantas inquietudes y tanta magia. Sin duda, esto hubiera sido mucho más aburrido sin ellos. Agradezco también a la Escuela Kodokai, que me ha llenado de inspiración y me ha ayudado a estar en forma durante estos años.

Estoy profundamente agradecido a mis padres, por darme una vida feliz y una buena educación, y por la confianza que siempre han depositado en mí. A Marta, por hacer que la vida sea más divertida, y a mi tía Nena, que siempre estuvo a nuestro lado.

Y, por supuesto, a Raquel y a mi pijama azul. Gracias por existir y por quererme tanto.

Finalmente, agradezco al MICINN por el apoyo económico prestado.

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Chapter 1

Introduction

General Relativity, formulated by Einstein in 1915 [54], is up to the present date the most accurate theory to describe gravitational physics. Roughly speaking, this theory establishes that space, time and gravitation are all of them aspects of a unique structure: the spacetime, a four dimensional manifold whose geometry is closely related to its matter contents via the Einstein field equations. One of the most striking consequences of General Relativity is the existence of *black holes*, that is, spacetime regions from which no signal can be seen by an observer located infinitely far from the matter sources. Black holes in the universe are expected to arise as the final state of gravitational collapse of sufficiently massive objects, such as massive stars, as the works by Chandrasekhar, Landau and Oppenheimer and Volkoff [35] already suggested in the decade of the 1930's. Despite the fact that many astronomical observations give strong indication that black holes really exist in nature, a definitive experimental proof of their existence is still lacking.

Although black holes arose first as theoretical predictions of General Relativity, its modern theory was developed in the mid-sixties largely in response to the astronomical discovery of highly energetic and compact objects. During these years the works of Hawking and Penrose [95] showed that singularities (i.e. "points" where the fundamental geometrical quantities are not well-defined) are commonplace in General Relativity, in particular in the interior of black holes. Singularities have the potential danger of breaking the predictability power of a theory because basically anything can happen once a singularity is visible. However, for the singularities inside black holes the situation is not nearly as bad, because, in this case, the singularity is not visible from infinity and hence the predictability capacity of the observers lying outside the black hole region remains unaffected. This fact led Penrose to conjecture that naked singularities (i.e. singularities which do not lie inside a black hole) cannot occur in any reasonable physical situation [96]. This conjecture, known as the *cosmic censorship hypothesis*, protects the distant observers from the lack of predictability that oc-

curs in the presence of singularities. Whether this conjecture is true or not is at present largely unknown (see [113] for an account of the situation in the late 90's). Rigorous results are known only in spherical symmetry, where the conjecture has been proven for several matter models [38, 50]. In any case, the validity of (some form) of cosmic censorship implies that black holes are the generic end state of gravitational collapse, and hence fundamental objects in the universe.

Of particular importance is the understanding of equilibrium configurations of black holes. The *uniqueness theorems for static and stationary black holes*, which are considered one of the cornerstones of the theory of black holes, also appeared during the sixties mainly motivated by the early work of Israel [71]. These theorems assert that, given a matter model (for example vacuum), a static or a stationary black hole spacetime belongs necessarily to a specific class of spacetimes (in the vacuum case, they are Schwarzschild in the static regime and Kerr for the stationary case) which are univocally characterized by a few parameters that describe the fundamental properties of the black hole (for vacuum these parameters are the mass and the angular momentum of the black hole). Since, from physical principles, it is expected that astronomical objects which collapse into a black hole will eventually settle down to a stationary state, the black hole uniqueness theorems imply that the final state of a generic gravitational collapse (assuming that cosmic censorship holds) can be described by a very simple spacetime geometry characterized by a few parameters like the total mass, the electric charge or the angular momentum of the collapsing astronomical object (or, more precisely, the amount of these physical quantities which is kept by the collapsing object and does not get radiated away during the process). The resulting spacetime is therefore independent of any other of the properties of the collapsing system (like shape, composition, etc.). This type of result was, somewhat pompously, named *"no hair"* theorems for black holes by Wheeler [103]. In 1973 Penrose [97] invoked cosmic censorship and the no hair theorems to deduce an inequality which imposes a lower bound for the total mass of a spacetime in terms of the area of the *event horizon* (i.e. the boundary) of the black hole which forms during the gravitational collapse. This conjecture is known as the *Penrose inequality*.

The Penrose inequality, like the cosmic censorship conjecture on which it is based, has been proven only in a few particular cases. Both conjectures therefore remain, up to now, wide open. One of the intrinsic difficulties for their proof is that black holes impose, by its very definition (see e.g. Chapter 12 of [112]), very strong global conditions on a spacetime. From an evolutive point if view,

these objects are of teleological nature because a complete knowledge of the future is needed to even know if a black hole forms. Determining the future of an initial configuration (i.e. the metric and its first time derivative on a spacelike hypersurface) requires solving the spacetime field equations (either analytical or numerically) with such initial data. The Einstein field equations are non-linear partial differential equations, so determining the long time behavior of its solutions is an extremely difficult problem. In general, the results that can be obtained from present day technology do not give information on the global structure of the solutions and, therefore, they do not allow to study black holes in an evolutive setting. As a consequence, the concept of black hole is not very useful in this situation because, what does it mean that an initial data set represents a black hole? Since the concept of black hole is central in gravitation, it has turned out to be necessary to replace this global notion by a more local one that, on the one hand, can be studied in an evolutionary setting and, on the other, hopefully has something to do with the global concept of black hole. The objects that serve this purpose are the so-called *trapped surfaces*, which are, roughly speaking, compact surfaces without boundary for which the emanating null rays do not diverge (all the precise definitions will be given in Chapter 2). The reason for this bending of light "inwards" is the gravitational field and, therefore, these surfaces reveal the presence of an intense gravitational field. This is expected to indicate that a black hole will in fact form upon evolution. More precisely, under suitable energy conditions, the maximal Cauchy development of this initial data is known to be causal geodesically incomplete (this is the content of one of the versions of the singularity theorems, see [105] for a review). *If cosmic censorship holds*, then a black hole will form. Moreover, it is known that in any black hole spacetime the subclass of trapped surfaces called *weakly trapped surfaces* and *weakly outer trapped surface* lie inside the black hole (see e.g. chapter 9.2 of [65] and chapter 12.2 of [112]), and so they give an indication of where the back hole event horizon should be in the initial data (if it forms at all). In fact, the substitution of the concept of black hole by the concept of trapped surface is so common that one terminology has replaced the other, and scientists talk about black hole collision, of black hole-neutron star mergers to refer to evolutions involving trapped surfaces. However, it should be kept in mind that both concepts are completely different a priori.

In the context of the Penrose inequality, the fact that, under cosmic censorship, weakly outer trapped surfaces lie inside the black hole was used by Penrose to replace the area of the event horizon by the area of weakly outer trapped sur-

faces to produce inequalities which, although motivated by the expected global structure of the spacetime that forms, can be formulated directly on the given initial data in a manner completely independent of its evolution. A particular case of weakly outer trapped surfaces, the so-called *marginally outer trapped surfaces (MOTS)* (defined as compact surfaces without boundary with vanishing outer null expansion θ^+), are widely considered as the best quasi-local replacements for the event horizon. From what it has been said, it is clear that proving that these surfaces can replace black holes is basically the same as proving the validity of cosmic censorship, which is beyond present day knowledge. The advantage of seeing the problem from this perspective is that it allows for simpler questions that can perhaps be solved. One such question is the Penrose inequality already mentioned. Another one has to do with static and stationary situations. One might think that, involving no evolution at all, it should be clear that black holes, event horizons and marginally outer trapped surfaces are essentially the same in an equilibrium configuration. However, although certainly plausible, very little is known about the validity of this expectation.

The aim of this thesis is precisely to study the properties of trapped surfaces in spacetimes with symmetries and their possible relation with the theory of black holes. Even this more modest goal is vast. We will concentrate on one aspect of this possible equivalence, namely *whether the static black hole uniqueness theorems extend to static spacetimes containing MOTS*. The main result of this thesis states that this question has an affirmative answer, under suitable conditions on the spacetime. To solve this question we will have to analyze in depth the properties of MOTS and weakly outer trapped surfaces in spacetimes with symmetries, and this will produce a number of results which are, hopefully, of independent interest. This study will naturally lead us to consider a second question, namely to study the Penrose inequality in static initial data sets which are not time-symmetric. Our main result here is the discovery of a counterexample of a version of the Penrose inequality that was proposed by Bray and Khuri [20] not long ago. It is worth to mention that most of the results we will obtain in this thesis do not use the Einstein field equations and, consequently, they are also valid in any gravitational theory of gravitation in four dimensions.

In the investigations on stationary and static spacetimes there has been a tendency over the years of reducing the amount of global assumptions in time to a minimum. This is in agreement with the idea behind cosmic censorship of understanding the global properties as a consequence of the evolution. This trend has been particularly noticeable in black hole uniqueness theorems, where several

conditions can be used to capture the notion of black hole (see e.g. Theorem 2.4.2 in Chapter 2). In this thesis, we will follow this general tendency and work directly on slabs of spacetimes containing suitable spacelike hypersurfaces or, whenever possible, directly at the initial data level, without assuming the existence of a spacetime where it is embedded. It should be remarked that the second setting is more general than the former one. Indeed, in some circumstances the existence of such a spacetime can be proven, for example by using the notion of Killing development (see [13] and Chapter 4) or by using well-posedness of the Cauchy problem and suitable evolution equations for the Killing vector [46]. The former, however, fails at fixed points of the static isometry and the second requires specific matter models, not just energy inequalities as we will assume. Nevertheless, although most of the results of this thesis will be obtained at the initial data level, we will need to invoke the existence of a spacetime to complete the proof of the uniqueness result (we emphasize however, that no global assumption in time is made in that case either). We will also try to make clear which is the difficulty that arises when one attempts to prove this result directly at the initial data level.

The results obtained in this thesis constitute, in our opinion, a step forward in our understanding of how black holes evolve. Regarding the problem of establishing a rigorous relationship between black holes and trapped surfaces, the main result of this thesis (Theorem 5.4.1) shows that, at least as far as uniqueness of static black holes is concerned, event horizons and MOTS do coincide. Our uniqueness result for static spacetimes containing MOTS is interesting also independently of its relationship with black holes. It proves that static configurations are indeed very rigid. This type of result has several implications. For instance, in any evolution of a collapsing system, it is expected that an equilibrium configuration is eventually reached. The uniqueness theorems of black holes are usually invoked to conclude that the spacetime is one of the stationary black holes compatible with the uniqueness theorem. However, this argument assumes implicitly that one has sufficient information on the spacetime to be able to apply the uniqueness theorems, which is far from obvious since the spacetime is being constructed during the evolution. In our setting, as long as the evolution has a MOTS on each time slice, if the spacetime reaches a static configuration, then it is unique. Related to this issue, it would be very interesting to know if these types of uniqueness results also hold in an approximate sense, i.e. if a spacetime is *nearly* static and contains a MOTS, then the spacetime is *nearly* unique. This problem is, of course, very difficult because it needs a suitable concept of “being

close to". In the particular case of the Kerr metric, there exists a notion of an initial data being close to Kerr [7] which is based on a suitable characterization of this spacetime [81]. This closeness notion is defined for initial data sets without boundary and has been extended to manifolds with boundary under certain circumstances [8]. It would be of interest to extend it to the case with a non-empty boundary which is a MOTS.

The static uniqueness result for MOTS is only a first step in this subject. Future work should try to extend this result to the stationary setting. The problem is, however, considerably more difficult because the techniques known at present to prove uniqueness of stationary black holes are much less developed than those for proving uniqueness of static black holes. Assuming however, that the spacetime is axially symmetric (besides being stationary) simplifies the black hole uniqueness proof considerably (the problem becomes essentially a uniqueness proof for a boundary value problem of a non-linear elliptic system on a domain in the Euclidean plane, see [67]). The next natural step would be to try and extend this uniqueness result to a setting where the black hole is replaced by a MOTS. The only result we prove in this thesis in the stationary (non-static) setting involves MOTS lying in the closure of the exterior region where the Killing is timelike. We show that in this case the MOTS cannot penetrate into the timelike exterior domain (see Theorem 3.4.10).

In the remaining of this Introduction, we will try to give a general idea of the structure of the thesis and to discuss its main results.

In rough terms, the typical structure of static black holes uniqueness theorems is the following:

Let $(M, g^{(4)})$ *be a static solution of the Einstein equations for a given matter model (for example vacuum) which describes a black hole. Then* $(M, g^{(4)})$ *belongs necessarily to a specific class of spacetimes which are univocally characterized by a number of parameters that can be measured at infinity (in the case of vacuum, the spacetime is necessarily Schwarzschild and the corresponding parameter is the total mass of the black hole).*

There exist static black hole uniqueness theorems for several matter models, such as vacuum ([71], [89], [100], [23], [39]), electro-vacuum ([72], [90], [108], [102], [109], [84], [40], [45]) and Einstein-Maxwell dilaton ([85], [83]). As we will describe in more detail in Chapter 2 the most powerful method for proving these results

is the so called *doubling method*, invented by Bunting and Masood-ul-Alam [23] to show uniqueness in the vacuum case. This method requires the existence of a complete spacelike hypersurface Σ containing an exterior, asymptotically flat, region Σ^{ext} such that the Killing is timelike on Σ^{ext} and the topological boundary $\partial^{top}\Sigma^{ext}$ is an embedded, compact and non-empty topological manifold. In static spacetimes, the condition that $(M, g^{(4)})$ is a black hole can be translated into the existence of such a hypersurface Σ. In this setting, the topological boundary $\partial^{top}\Sigma^{ext}$ corresponds to the intersection of the boundary of the domain of outer communications (i.e. the region outside both the black hole and the white hole) and Σ. This equivalence, however, is not strict due to the potential presence of non-embedded Killing prehorizons, which would give rise to boundaries $\partial^{top}\Sigma^{ext}$ which are non-embedded. This issue is important and will be discussed in detail below. We can however, ignore this subtlety for the purpose of this Introduction.

The type of uniqueness result we are interested in this thesis is of the form:

Let $(M, g^{(4)})$ be a static solution of the Einstein equations for a given matter model. Suppose that M possesses a spacelike hypersurface Σ which contains a MOTS. Then, $(M, g^{(4)})$ belongs to the class of spacetimes established by the uniqueness theorem for static black holes for the corresponding matter model.

The first result in this direction was given by Miao in 2005 [88], who extended the uniqueness theorems for vacuum static black holes to the case of asymptotically flat and time-symmetric slices Σ which contain a minimal compact boundary (it is important to note that for time-symmetric initial data, a surface is a MOTS if and only if it is a compact minimal surface). In this way, Miao was able to relax the condition of a time-symmetric slice Σ having a compact topological boundary $\partial^{top}\Sigma$ where the Killing vector vanishes to simply containing a compact minimal boundary. Miao's uniqueness result is indeed a generalization of the static uniqueness theorem of Bunting and Masood-ul-Alam because the static vacuum field equations imply in the time-symmetric case that the boundary $\partial^{top}\Sigma^{ext}$ is necessarily a totally geodesic surface, which is more restrictive than being a minimal surface.

Miao's result is fundamentally a uniqueness result. However, one of the key ingredients in its proof consists in showing that no minimal surface can penetrate into the exterior timelike region Σ^{ext}. As a consequence, Miao's theorem can also be viewed as a confinement result for minimal surfaces. As a consequence, one can think of extending Miao's result in three different directions: Firstly, to

allow for other matter models. Secondly, to work with arbitrary slices and not just time-symmetric ones. This is important in order to be able to incorporate so-called degenerate Killing horizons into the problem. Obviously, in the general case minimal surfaces are no longer suitable and MOTS should be considered. And finally, try to make the confinement part of the statement as local as possible and relax the condition of asymptotic flatness to the existence of suitable exterior barrier. To that aim it is necessary a proper understanding of the properties of MOTS and weakly outer trapped surfaces in static spacetimes (or more general, if possible).

For simplicity, let us restrict to the asymptotically flat case for the purpose of the Introduction. Consider a spacelike hypersurface Σ containing an asymptotically flat end Σ_0^∞. In what follows, let λ be minus the squared norm of the static Killing $\vec{\xi}$. So, $\lambda > 0$ means that $\vec{\xi}$ is timelike. Staticity and asymptotic flatness mean that this Killing vector is timelike at infinity. Thus, it makes sense to define $\{\lambda > 0\}^{ext}$ as the connected component of $\{\lambda > 0\}$ which contains the asymptotically flat end Σ_0^∞ (the set Σ^{ext} in the Masood-ul-Alam doubling method is precisely $\{\lambda > 0\}^{ext}$). Since we want to prove the expectation that MOTS and spacelike sections of the event horizon coincide in static spacetimes, we will firstly try to ensure that no MOTS can penetrate into $\{\lambda > 0\}^{ext}$. This result will generalize Miao's theorem as a confinement result and will extend the well-known confinement result of MOTS inside the black hole region (c.f. Proposition 12.2.4 in [112])) to the initial data level. The main tool which will allow us to prove this result is a recent theorem by Andersson and Metzger [4] on the existence, uniqueness and regularity of the outermost MOTS on a given spacelike hypersurface. This theorem, which will be essential in many places in this thesis, requires working with trapped surfaces which are *bounding*, in the sense that they are boundaries of suitable regions (see Definition 2.2.26). Another important ingredient for our confinement result will be a thorough study of the causal character that the Killing vector is allowed to have on the outermost MOTS (or, more, generally on stable or strictly stable MOTS – all these concepts will be defined below –). For the case of *weakly trapped surfaces* (which are defined by a more restrictive condition than weakly outer trapped surfaces), it was proven in [82] that no weakly trapped surface can lie in the region where the Killing vector is timelike provided its mean curvature vector does not vanish identically. Furthermore, similar restrictions were also obtained for other types of symmetries, such as conformal Killing vectors (see also [107] for analogous results in spacetimes with vanishing curvature invariants).

Our main idea to obtain restrictions on the Killing vector on an outermost MOTS S consists on a geometrical construction [24] whereby S is moved first to the past along the integral lines of the Killing vector and then back to Σ along the outer null geodesics orthogonal to this newly constructed surface, producing a new weakly outer trapped surface S', provided the null energy condition (NEC) is satisfied in the spacetime. If the Killing field $\vec{\xi}$ is timelike anywhere on S then we show that S' lies partially outside S, which is a contradiction with the outermost property of S. This simple idea will be central in this thesis and will be extended in several directions. In particular, we will generalize the geometric construction to the case of general vector fields $\vec{\xi}$, not just Killing vectors. To ensure that S' is weakly outer trapped in this setting we will need to obtain an explicit expression for the first variation of the outer null expansion θ^+ along $\vec{\xi}$ in terms of the so called *deformation tensor* of the metric along $\vec{\xi}$ (Proposition 3.3.1). This will allow us to obtain results for other types of symmetries, such as homotheties and conformal Killing vectors, which are relevant in many physical situations of interest (e.g. the Friedmann-Lemaître-Robertson-Walker cosmological models). Another relevant generalization involves analyzing the infinitesimal version of the geometric construction. As we will see, the infinitesimal construction is closely related to the stability properties of the the first variation of θ^+ along Σ on a MOTS S. This first variation defines a linear elliptic second order differential operator [3] for which elliptic theory results can be applied. It turns out that exploiting such results (in particular, the maximum principle for elliptic operators) the conclusions of the geometric construction can be sharpened considerably and also extended to more general MOTS such as stable and strictly stable ones. (Theorem 3.4.2 and Corollaries 3.4.3 and 3.4.4).

As an explicit application of these results, we will show that stable MOTS cannot exist in any slice of a large class of Friedmann-Lemaître-Robertson-Walker cosmological models. This class includes all classic models of matter and radiation dominated eras and also those models with accelerated expansion which satisfy the NEC (Theorem 3.4.6). Remarkably, the geometric construction is more powerful than the elliptic methods in some specific cases. We will find an interesting situation where this is the case when dealing with homotheties (including Killing vectors) on outermost MOTS (Theorem 3.4.8). This will allow us to prove a result (Theorem 3.4.10) which asserts that, as long as the spacetime satisfies the NEC, a Killing vector or homothety cannot be timelike anywhere on a bounding weakly outer trapped surface whose exterior lies in a region where the Killing vector is timelike.

Another case when the elliptic theory cannot be applied and we resort to the geometric procedure deals with situations when one cannot ensure that the newly constructed surface S' is weakly outer trapped. However, it can still occur that the portion of S' which lies in the exterior of S has $\theta^+ \leq 0$. In this case, we can exploit a result by Kriele and Hayward [77] in order to construct a weakly outer trapped surface S'' outside both S and S' by smoothing outwards the corner where they intersect. This will provide us with additional results of interest (Theorems 3.5.2 and 3.5.4). All these results have been published in [26] and [27] and will be presented in Chapter 3.

From then on, we will concentrate exclusively on *static* spacetimes. Chapter 4 is devoted to extending Miao's result as a confinement result. Since in this chapter we will work exclusively at the initial data level, we will begin by recalling the concept of a *static Killing initial data (static KID)*, (which corresponds to the data and equations one induces on any spacelike hypersurface embedded on a static spacetime, but viewed as an abstract object on its own, independently of the existence of any embedding into a spacetime). It will be useful to introduce two scalars I_1, I_2 which correspond to the invariants of the *Killing form* (or Papapetrou field) of the static Killing vector $\vec{\xi}$. It turns out that I_2 always vanishes due to staticity and that I_1 is constant on arc-connected components of $\partial^{top}\{\lambda > 0\}$ and negative on the arc-connected components which contains at least a fixed point (Lemma 4.3.6). Fixed points are initial data translations of spacetime points where the Killing vector vanishes and, since I_1 turns out to be closely related to the surface gravity of the Killing horizons, this result extends a well-known result by Boyer [17] on the structure of Killing horizons to the initial data level.

The general strategy to prove our confinement result for MOTS is to use a contradiction argument. We will assume that a MOTS can penetrate in the exterior timelike region. By passing to the outermost MOTS S we will find that the topological boundary of $\partial^{top}\{\lambda > 0\}^{ext}$ must intersect both the interior and the exterior of S. It we knew that $\partial^{top}\{\lambda > 0\}^{ext}$ is a bounding MOTS, then we could get a contradiction essentially by smoothing outwards (via the Kriele and Hayward method) these two surfaces. However, it is not true that $\partial^{top}\{\lambda > 0\}^{ext}$ is a bounding MOTS in general. There are simple examples even in Kruskal where this property fails. The problem lies in the fact that $\partial^{top}\{\lambda > 0\}^{ext}$ can intersect both the black hole and the white hole event horizons (think of the Kruskal spacetime for definiteness) and then the boundary $\partial^{top}\{\lambda > 0\}^{ext}$ is, in general, not smooth on the bifurcation surface. To avoid this situations we need

to assume a condition which essentially imposes that $\partial^{top}\{\lambda > 0\}^{ext}$ intersects only the black hole or only the white hole region. Furthermore, the possibility of $\partial^{top}\{\lambda > 0\}^{ext}$ intersecting the white hole region must be removed to ensure that this smooth surface is in fact a MOTS and not a *past* MOTS. The precise statement of this final condition is given in points (i) and (ii) of Proposition 4.3.15, but the more intuitive idea above is sufficient for this Introduction. Since we will need to mention this condition below, we refer to it as ($\star$). In this way, in Proposition 4.3.15, we prove that every arc-connected component of $\partial^{top}\{\lambda > 0\}$ is an injectively immersed submanifold with $\theta^{+} = 0$. However, injectively immersed submanifolds may well not be embedded. Since, in order to find a contradiction we need to construct a bounding weakly outer trapped surface, and these are necessarily embedded, we need to care about proving that the injective immersion is an embedding (i.e. an homeomorphism with the induced topology in the image). In the case with $I_1 \neq 0$ this is easy. In the case of components with $I_1 = 0$ (so-called *degenerate* components), the problem is difficult and open. This issue is very closely related to the possibility that there may exist non-embedded Killing prehorizons in a static spacetime which has already been mentioned before. This problem, which has remained largely overlooked in the black hole uniqueness theory until very recently [41], is important and very interesting. However, it is beyond the scope of this thesis. For our purposes it is sufficient to assume an extra condition on degenerate components of $\partial^{top}\{\lambda > 0\}^{ext}$ which easily implies that they are embedded submanifolds. This condition is that every arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ with $I_1 = 0$ is topologically closed. This requirement will appear in all the main results in this thesis precisely in order to avoid dealing with the possibility of non-embedded Killing prehorizons. If one can eventually prove that such objects simply do not exist (as we expect), then this condition can simply be dropped in all the results below. Our main confinement result is given in Theorem 4.4.1. The results of Chapter 4 have been published in [24] and [25].

Theorem 4.4.1 leads directly to a uniqueness result (Theorem 5.1.1) which already generalizes Miao's result as a uniqueness statement. The idea of the uniqueness proof is to show that the presence of a MOTS boundary in an initial data set implies, under suitable conditions, that $\partial^{top}\{\lambda > 0\}^{ext}$ is a compact embedded surface *without boundary*. This is precisely the main hypothesis that is made in order to apply the doubling method of Bunting and Masood-ul-Alam. Thus, assuming that the matter model is such that static black hole uniqueness holds, then we can conclude uniqueness in the case with MOTS. The strategy is there-

fore to reduce the uniqueness theorem for MOTS to the uniqueness theorem for black holes. This idea is in full agreement with our main theme of showing that MOTS and black holes are the same in a static situation.

Theorem 5.1.1 is, however, not fully satisfactory because it still requires condition ($\star$) on $\partial^{top}\{\lambda > 0\}^{ext}$. Since $\partial^{top}\{\lambda > 0\}^{ext}$ is a fundamental object in the doubling method, it would be preferable if no conditions are a priori imposed on it. Chapter 5 is devoted to obtaining a uniqueness result for static spacetimes containing weakly outer trapped surfaces with no a priori restrictions on $\partial^{top}\{\lambda > 0\}^{ext}$ (besides the condition on components with $I_1 = 0$ which we have already mentioned). In Chapter 4 the fact that $\partial^{top}\{\lambda > 0\}^{ext}$ is closed (i.e. compact and without boundary) is proven as a consequence of its smoothness. However, when condition ($\star$) is dropped, we know that $\partial^{top}\{\lambda > 0\}^{ext}$ is not smooth in general, and in principle, it may have a non-empty manifold boundary. Therefore, we will need a better understanding of the structure of the set $\partial^{top}\{\lambda > 0\}$ when ($\star$) is not assumed. In this case, our methods of Chapter 4 do not work and we will be forced to invoke the existence of a spacetime where the initial data set is embedded. By exploiting a construction by Rácz and Wald in [98] we show that, in an embedded static KID, the set $\partial^{top}\{\lambda > 0\}$ is a finite union of smooth, compact and embedded surfaces, possibly with boundary. Moreover, at least one of the two null expansions θ^+ or θ^- vanishes identically on each one of these surfaces (Proposition 5.3.1). With this result at hand we then prove that the set $\partial^{top}\{\lambda > 0\}^{ext}$ coincides with the outermost bounding MOTS (Theorem 5.3.3) provided the spacetime satisfies the NEC and that the *past weakly outer trapped region* T^- is included in the *weakly outer trapped region* T^+. It may seem that the condition $T^- \subset T^+$ is very similar to ($\star$): In some sense, both try to avoid that the slice intersects first the white hole horizon when moving from the outside. However, it is important to remark that T^+ and T^- have a priori nothing to do with Killing horizons and that the condition $T^- \subset T^+$ is not a condition directly on $\partial^{top}\{\lambda > 0\}^{ext}$. Our main uniqueness theorem is hence Theorem 5.4.1, which states that, under reasonable hypotheses, MOTS and spacelike sections of Killing horizons do coincide in static spacetimes. If the static spacetime is a black hole (in the global sense) then the event horizon is a Killing horizon. This shows the equivalence between MOTS and (spacelike sections of) the event horizon in the static setting.

The last part of this thesis is devoted to the study of the Penrose inequality in initial data sets which are not time-symmetric. The standard version of the Penrose inequality bounds the ADM mass of the spacetime in terms of the small-

est area of all surfaces which enclose the outermost MOTS. The huge problem in proving this inequality has led several authors to propose more general and simpler looking versions of the Penrose inequality (see [80] for a review). In particular, in a recent proposal by Bray and Khuri [20], a Penrose inequality has been conjectured in terms of the area of so-called outermost *generalized apparent horizon* in a given asymptotically flat initial data set. Generalized apparent horizons are more general than weakly outer trapped surfaces and have interesting analytic and geometric properties. The Penrose inequality conjectured by Bray and Khuri reads

$$M_{ADM} \geq \sqrt{\frac{|S_{out}|}{16\pi}}, \tag{1.1}$$

where M_{ADM} is the total ADM mass of a given slice and $|S_{out}|$ is the area of the outermost generalized apparent horizon S_{out}. This new inequality has several appealing properties, like being invariant under time reversals, the fact that no minimal area enclosures are involved and that it implies the standard Penrose inequality. On the other hand, this version is not directly supported by any heuristic argument based on cosmic censorship, as the standard Penrose inequality. In fact, as a consequence of a theorem by Eichmair [53] on the existence, uniqueness and regularity of the outermost generalized apparent horizon, there exist slices in the Kruskal spacetimes (for which $\partial^{top}\{\lambda > 0\}^{ext}$ intersects both the black hole and the white hole event horizons), with the property that its outermost generalized apparent horizon lies, at least partially, inside the domain of outer communications. In Chapter 6 we present a counterexample of (1.1) precisely by studying this type of slices in the Kruskal spacetime.

The equations that define a generalized apparent horizon are non-linear elliptic PDE. Thus, we intend to determine properties of the solutions of these equations for slices sufficiently close to the time-symmetric slice of the Kruskal spacetime. Since the outermost generalized apparent horizon in the time-symmetric slice is the well-known bifurcation surface, we can exploit the implicit function theorem to show that any solution of the linearized equation for the generalized apparent horizon corresponds to the linearization of a solution of the non-linear problem (Proposition 6.2.2). With this existence result at hand, we find a generalized apparent horizon $\hat{S}$ which turns out to be located entirely inside the domain of outer communications and which has area larger than $16\pi M_{ADM}^2$, this violating (1.1). This would give a counterexample to the Bray and Khuri conjecture provided $\hat{S}$ is either the outermost generalized apparent horizon S_{out} or else, the latter has not smaller area than the former one. Finally, we will prove that the area of S_{out}

is, indeed, at least as large as the area of $\hat{S}$, which gives a counterexample to (1.1) (Theorem 6.1.1). It is important to remark that the existence of this counterexample does not invalidate the approach given by Bray and Khuri in [20] to prove the standard Penrose inequality but it does indicate that the emphasis must not be on generalized apparent horizons. This result has been published in [28] and [29].

Before going into our new results, we start with a preliminary chapter where the fundamental definitions and results required to understand this thesis are stated and briefly discussed. This chapter contains in particular, a detailed sketch of the Bunting and Masood-ul-Alam method to prove uniqueness of electro-vacuum static black holes. We have preferred to collect all the preliminary material in one chapter to facilitate the reading of the thesis. We have also found it convenient to include two mathematical appendices. One where some well-known definitions of manifolds with boundary and topology are included (Appendix A) and another one that collects a number of theorems in mathematical analysis (Appendix B) which are used as tools in the main text.

Chapter 2

Preliminaries

2.1 Basic elements in a geometric theory of gravity

The fundamental concept in any geometric theory of gravity is that of spacetime. A **spacetime** is a connected n-dimensional smooth differentiable manifold M without boundary endowed with a Lorentzian metric $g^{(n)}$. All manifolds considered in this thesis will be Hausdorff. (see Appendix A for the definition). A Lorentzian metric is a metric with signature $(-,+,+,...,+)$. The covariant derivative associated with the Levi-Civita connection of $g^{(n)}$ will be denoted by $\nabla^{(n)}$ and the corresponding Riemann, Ricci and scalar curvature tensors will be denoted by $R^{(n)}_{\mu\nu\alpha\beta}$, $R^{(n)}_{\mu\nu}$ and $R^{(n)}$, respectively (where $\mu,\nu,\alpha,\beta = 0,...,n-1$). We follow the sign conventions of [112]. We will denote by $T_{\mathfrak{p}}M$ the tangent space to M at a point $\mathfrak{p} \in M$, by TM the tangent bundle to M (i.e. the collection of the tangent spaces at every point of M) and by $\mathfrak{X}(M)$ the set of smooth sections of TM (i.e. vector fields on M).

Definition 2.1.1 *According to the sign of its squared norm, a vector $\vec{v} \in T_{\mathfrak{p}}M$ is:*

- *Spacelike, if* $g^{(n)}_{\mu\nu} v^\mu v^\nu \Big|_{\mathfrak{p}} > 0$.
- *Timelike, if* $g^{(n)}_{\mu\nu} v^\mu v^\nu \Big|_{\mathfrak{p}} < 0$.
- *Null, if* $g^{(n)}_{\mu\nu} v^\mu v^\nu \Big|_{\mathfrak{p}} = 0$.
- *Causal, if* $g^{(n)}_{\mu\nu} v^\mu v^\nu \Big|_{\mathfrak{p}} \leq 0$.

Definition 2.1.2 *A spacetime $(M, g^{(n)})$ is* **time orientable** *if and only if there exists a vector field $\vec{u} \in \mathfrak{X}(M)$ which is timelike everywhere on M.*

Consider a time orientable spacetime $(M, g^{(n)})$. *A* **time orientation** *is a selection of a timelike vector field* $\vec{u}$ *which is declared to be future directed.*
A **time oriented** *spacetime is a time orientable spacetime after a time orientation has been selected.*

In a time oriented manifold, causal vectors can be classified in two types: future directed or past directed.

Definition 2.1.3 *Let* $(M, g^{(n)})$ *be a spacetime with time orientation* $\vec{u}$. *Then, a causal vector* $\vec{v} \in T_{\mathfrak{p}}M$ *is*

- *future directed if* $g^{(n)}_{\mu\nu} u^\mu v^\nu \Big|_{\mathfrak{p}} \leq 0$.
- *past directed if* $g^{(n)}_{\mu\nu} u^\mu v^\nu \Big|_{\mathfrak{p}} \geq 0$.

Throughout this thesis all spacetimes are oriented (see Definition A.6 in Appendix A) and time oriented.

General Relativity is a geometric theory of gravity in four dimensions in which the spacetime metric $g^{(4)}$ satisfies the Einstein field equations, which in geometrized units, $G = c = 1$ (where G is the Newton gravitational constant and c is the speed of light in vacuum), takes the form:

$$G^{(4)}_{\mu\nu} + \Lambda g^{(4)}_{\mu\nu} = 8\pi T_{\mu\nu}, \tag{2.1.1}$$

where $G^{(4)}_{\mu\nu}$ is the so-called Einstein tensor, $G^{(n)}_{\mu\nu} \equiv R^{(n)}_{\mu\nu} - \frac{1}{2} R^{(n)} g^{(n)}_{\mu\nu}$ (in n dimensions), Λ is the so-called cosmological constant and $T_{\mu\nu}$ is the stress-energy tensor which describes the matter contents of the spacetime. In such a framework, freely falling test bodies are assumed to travel along the causal (timelike for massive particles and null for massless particles) geodesics of the spacetime $(M, g^{(4)})$.

Due to general physical principles, it is expected that many dynamical processes tend to a stationary final state. Studying these stationary configurations is therefore an essential step for understanding any physical theory. This is the case, for example, in gravitational collapse processes in General Relativity which are expected to settle down to a stationary system. Since the fundamental object in gravity is the spacetime metric $g^{(4)}$, the existence of symmetries in the spacetime is expressed in terms of a group of isometries, that is, diffeomorphisms of the spacetime manifold M which leave the metric unchanged. The infinitesimal generator of the isometry group defines a so-called *Killing vector field*. Conversely,

a Killing vector field defines a local isometry, i.e. a local group of diffeomorphisms, each of which is an isometry of $(M, g^{(4)})$. If the Killing vector field is complete then the local group is, in fact, a global group of isometries (or, simply, an isometry). Throughout this thesis, we will mainly work at the local level without assuming that the Killing vector fields are complete, unless otherwise stated. More precisely, consider a spacetime $(M, g^{(4)})$ and a vector field $\vec{\xi} \in \mathfrak{X}(M)$. The Lie derivative $\mathcal{L}_{\vec{\xi}}\, g^{(4)}_{\mu\nu}$ describes how the metric is deformed along the local group of diffeomorphisms generated by $\vec{\xi}$. We thus define the **metric deformation tensor** associated to $\vec{\xi}$, or simply deformation tensor, as

$$a_{\mu\nu}(\vec{\xi}\,) \equiv \mathcal{L}_{\vec{\xi}}\, g^{(4)}_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu, \tag{2.1.2}$$

where, throughout this thesis, ∇ will denote the covariant derivative of $g^{(4)}$. If $a_{\mu\nu}(\vec{\xi}\,) = 0$, then the vector field $\vec{\xi}$ is a **Killing vector field** or simply a Killing vector.
If the Killing field is timelike on some non-empty set, then the spacetime is called **stationary**. If, furthermore, the Killing field is integrable, i.e.

$$\xi_{[\mu} \nabla_\nu \xi_{\alpha]} = 0 \tag{2.1.3}$$

where the square brackets denote anti-symmetrization, then the spacetime is called **static**.
Other important types of isometries are the following. If the Killing field is spacelike and the isometry group generated is $U(1)$, then the spacetime has a **cyclic symmetry**. If, furthermore, there exists a regular axis of symmetry, then the spacetime is **axisymmetric**. If the isometry group is $SO(3)$ with orbits being spacelike 2-spheres (or points), then the spacetime is **spherically symmetric**.
Other special forms of $a_{\mu\nu}(\vec{\xi}\,)$ define special types of vectors which are also interesting. In particular, $a_{\mu\nu}(\vec{\xi}\,) = 2\phi g^{(4)}_{\mu\nu}$ (with ϕ being a scalar function) defines a **conformal Killing vector** and $a_{\mu\nu}(\vec{\xi}\,) = 2C g^{(4)}_{\mu\nu}$ (with C being a constant) corresponds to a **homothety**.

Regarding the matter contents of the spacetime, represented by $T_{\mu\nu}$, we will not assume a priori any specific matter model, such as vacuum, electro-vacuum, perfect fluid, etc. However, we will often restrict the class of models in such a way that various types of so-called energy conditions are satisfied (c.f. Chapter 9.2 in [112]). These are inequalities involving $T_{\mu\nu}$ acting on certain causal vectors and are satisfied by most physically reasonable matter models. In fact, since in General

Relativity without cosmological constant, the Einstein equations impose $G^{(4)}_{\mu\nu} = 8\pi T_{\mu\nu}$, these conditions can be stated directly in terms of the Einstein tensor. We choose to define the energy conditions directly in terms of $G^{(4)}_{\mu\nu}$. This is preferable because then all our results hold in any geometric theory of gravity independently of whether the Einstein field equations hold or not. Obviously, these inequalities are truly energy conditions only in specific theories as, for instance, General Relativity with $\Lambda = 0$. Throughout this thesis, we will often need to impose the so-called *null energy condition (NEC)*.

Definition 2.1.4 *A spacetime $(M, g^{(4)})$ satisfies the* **null energy condition (NEC)** *if the Einstein tensor $G^{(4)}_{\mu\nu}$ satisfies $G^{(4)}_{\mu\nu} k^\mu k^\nu |_{\mathfrak{p}} \geq 0$ for any null vector $\vec{k} \in T_{\mathfrak{p}} M$ and all $\mathfrak{p} \in M$.*

Other usual energy conditions are the *weak energy condition* and the *dominant energy condition (DEC)*.

Definition 2.1.5 *A spacetime $(M, g^{(4)})$ satisfies the* **weak energy condition** *if the Einstein tensor $G^{(4)}_{\mu\nu}$ satisfies that $G^{(4)}_{\mu\nu} t^\mu t^\nu |_{\mathfrak{p}} \geq 0$ for any timelike vector $\vec{t} \in T_{\mathfrak{p}} M$ and all $\mathfrak{p} \in M$.*

Definition 2.1.6 *A spacetime $(M, g^{(4)})$ satisfies the* **dominant energy condition (DEC)** *if the Einstein tensor $G^{(4)}_{\mu\nu}$ satisfies that $-G^{(4)\nu}{}_{\mu} t^\mu |_{\mathfrak{p}}$ is a future directed causal vector for any future directed timelike vector $\vec{t} \in T_{\mathfrak{p}} M$ and all $\mathfrak{p} \in M$.*

Remark. Obviously, the DEC implies the NEC. □

2.2 Geometry of surfaces in Lorentzian spaces

2.2.1 Definitions

In this subsection we will motivate and introduce several types of surfaces, such as trapped surfaces and marginally outer trapped surfaces, that will play an important role in this thesis. We will also discuss several relevant known results concerning them. For an extensive classification of surfaces in Lorentzian spaces, see [106]. Let us begin with some previous definitions and notation.

In what follows, M and Σ are two smooth differentiable manifolds, Σ possibly with boundary, with dimensions n and s, respectively, satisfying $n \geq s$.

Definition 2.2.1 *Let $\Phi : \Sigma \to M$ be a smooth map between Σ and M. Then Φ is an* **immersion** *if its differential has maximum rank (i.e. rank$(\Phi) = s$) at every point.*

The set $\Phi(\Sigma)$ is then said to be *immersed* in M. However $\Phi(\Sigma)$ can fail to be a manifold because it can intersect itself.
To avoid self-intersections, one has to consider *injective immersions*. In fact, we will say that $\Phi(\Sigma)$ is a **submanifold** of M if Σ is injectively immersed in M. All immersions considered in this thesis will be submanifolds. For simplicity, and since no confusion usually arises, we will frequently denote by the same symbol (Σ in this case) both the manifold Σ (as an abstract manifold) and $\Phi(\Sigma)$ (as a submanifold). Similarly, and unless otherwise stated, we will use the same convention for contravariant tensors. More specifically, a contravariant tensor defined on Σ and pushed-forward to $\Phi(\Sigma)$ will be usually denoted by the same symbol. Notice however that $\Phi(\Sigma)$ admits two topologies which are in general different: the induced topology as a subset of M and the manifold topology defined by Φ from Σ. When referring to topological concepts in injectively immersed submanifolds we will always use the subset topology unless otherwise stated.

Next, we will define the first and the second fundamental forms of a submanifold.

Definition 2.2.2 *Consider a smooth manifold M endowed with a metric $g^{(n)}$ and let Σ be a submanifold of M. Then, the* **first fundamental form** *of Σ is the tensor field g on Σ defined as*

$$g = \Phi^* \left(g^{(n)} \right),$$

where Φ^* denotes the pull-back of the injective immersion $\Phi : \Sigma \to M$.

According to the algebraic properties of its first fundamental form, a submanifold can be classified as follows.

Definition 2.2.3 *A submanifold Σ of a spacetime M is:*

- **Spacelike** *if g is non-degenerate and positive definite.*
- **Timelike** *if g is non-degenerate and non-positive definite.*
- **Null** *if g is degenerate.*

The following result is straightforward and well-known (see e.g. [94])

Proposition 2.2.4 *Let Σ be a submanifold of M. Then, the first fundamental form g of Σ is non-degenerate (and, therefore, a metric) at a point $\mathfrak{p} \in \Sigma$ if and only if*

$$T_{\mathfrak{p}}M = T_{\mathfrak{p}}\Sigma \oplus (T_{\mathfrak{p}}\Sigma)^{\perp}, \tag{2.2.1}$$

where $(T_{\mathfrak{p}}\Sigma)^{\perp}$ denotes the set of normal vectors to Σ at $\mathfrak{p}$.

We will denote $(T_{\mathfrak{p}}M)^{\perp}$ by $N_{\mathfrak{p}}M$ and we will call this set the normal space to Σ at $\mathfrak{p}$. The collection of all normal spaces forms a vector bundle over Σ which is called the normal bundle and is denoted by $N\Sigma$. From now on, unless otherwise stated, we will only consider submanifolds satisfying (2.2.1) at every point. Let us denote by ∇^{Σ} the covariant derivative associated with g.

Next, consider two arbitrary vectors $\vec{X}, \vec{Y} \in \mathfrak{X}(\Sigma)$. According to (2.2.1), the derivative $\nabla^{(n)}_{\vec{X}}\vec{Y}$, as a vector on TM, can be split according to

$$\nabla^{(n)}_{\vec{X}}\vec{Y} = \left(\nabla^{(n)}_{\vec{X}}\vec{Y}\right)^{T} + \left(\nabla^{(n)}_{\vec{X}}\vec{Y}\right)^{\perp},$$

where the superindices T and $\perp$ denote the tangential and normal parts with respect to Σ. The following is an important result in the theory of submanifolds [94].

Theorem 2.2.5 *With the notation above, we have*

$$\left(\nabla^{(n)}_{\vec{X}}\vec{Y}\right)^{T} = \nabla^{\Sigma}_{\vec{X}}\vec{Y}.$$

The extrinsic geometry of the submanifold is encoded in its second fundamental form.

Definition 2.2.6 *The* **second fundamental form vector** *$\vec{K}$ of Σ in M is a symmetric linear map $\vec{K} : \mathfrak{X}(\Sigma) \times \mathfrak{X}(\Sigma) \rightarrow N\Sigma$ defined by*

$$\vec{K}(\vec{X}, \vec{Y}) = -\left(\nabla^{(n)}_{\vec{X}}\vec{Y}\right)^{\perp},$$

for all $\vec{X}, \vec{Y} \in \mathfrak{X}(\Sigma)$.

Remark. Our sign convention is such that the second fundamental form vector of a 2-sphere in the Euclidean 3-space points outwards. □

Definition 2.2.7 *The* **mean curvature vector** *of Σ in M is defined as $\vec{H} \equiv tr_{\Sigma}\vec{K}$ (where tr_{Σ} denotes the trace with the induced metric g on $T_{\mathfrak{p}}\Sigma$ for any $\mathfrak{p} \in \Sigma$).*

Definition 2.2.8 *We will define an* **embedding** Φ *as an injective immersion such that* $\Phi : \Sigma \rightarrow \Phi(\Sigma)$ *is an homeomorphism with the topology on* $\Phi(\Sigma)$ *induced from* M*. The image* $\Phi(\Sigma)$ *will be called an embedded submanifold.*

Definition 2.2.9 *A* **surface** S *is a smooth, orientable, codimension two, embedded submanifold of* M *with positive definite first fundamental form* γ*.*

From now on we will focus on 4-dimensional spacetimes $(M, g^{(4)})$. For a surface $S \subset M$ we have the following result.

Lemma 2.2.10 *The normal bundle of* S *admits two vector fields* $\left\{\vec{l}_+, \vec{l}_-\right\}$ *which are null and future directed everywhere, and which form a basis of* NS *in* TM *at every point* $\mathfrak{p} \in S$*.*

Proof. Let $\mathfrak{p} \in S$ and $(U_\alpha, \varphi_\alpha)$ be any chart at $\mathfrak{p}$ belonging to the positively oriented atlas of M. Let us define $\{\vec{l}_+^{U_\alpha}, \vec{l}_-^{U_\alpha}\}$ as the solution of the set of equations

$$\begin{aligned} &g^{(4)}(\vec{l}_\pm^{U_\alpha}, \vec{e}_A)\Big|_{\mathfrak{p}} = 0, \qquad && g^{(4)}(\vec{l}_\pm^{U_\alpha}, \vec{l}_\pm^{U_\alpha})\Big|_{\mathfrak{p}} = 0, \\ &g^{(4)}(\vec{l}_+^{U_\alpha}, \vec{l}_-^{U_\alpha})\Big|_{\mathfrak{p}} = -2, \qquad && g^{(4)}(\vec{l}_+^{U_\alpha}, \vec{u})\Big|_{\mathfrak{p}} = -1, \\ & && \eta^{(4)}(\vec{l}_-^{U_\alpha}, \vec{l}_+^{U_\alpha}, \vec{e}_1, \vec{e}_2)\Big|_{\mathfrak{p}} > 0. \end{aligned} \tag{2.2.2}$$

where the vectors $\{\vec{e}_A\}$ $(A = 1, 2)$ are the coordinate basis in U_α, $\vec{u}$ is the timelike vector which defines the time-orientation for the spacetime and $\eta^{(4)}$ is the volume form of $(M, g^{(4)})$. It is immediate to check that $\{\vec{l}_+^{U_\alpha}, \vec{l}_-^{U_\alpha}\}$ exists and is unique. The last equation is necessary in order to avoid the ambiguity $\vec{l}_+^{U_\alpha} \leftrightarrow \vec{l}_-^{U_\alpha}$ allowed by the previous four equations.

The set $\{\vec{l}_+^{U_\alpha}, \vec{l}_-^{U_\alpha}\}$ defines two vector fields if and only if this definition is independent of the chart. Select any other positively oriented chart (U_β, φ_β) at $\mathfrak{p}$. Let $\{\vec{e'}_1, \vec{e'}_2\}$ be the corresponding coordinate basis, which is related with $\{\vec{e}_1, \vec{e}_2\}$ by ${e'}_A^{\,\mu} = A^\mu_\nu e^\nu_A$ $(A, B = 1, 2)$, where A^μ_ν denotes the Jacobian. Since U_α and U_β belong to the positively oriented atlas, we have that $\det A > 0$ everywhere.

The first four equations in (2.2.2) force that either $\vec{l}_\pm^{U_\beta} = \vec{l}_\pm^{U_\alpha}$ or $\vec{l}_\pm^{U_\beta} = \vec{l}_\mp^{U_\alpha}$. However, the second possibility would imply

$$\eta^{(4)}(\vec{l}_-^{U_\beta}, \vec{l}_+^{U_\beta}, \vec{e'}_1, \vec{e'}_2)\Big|_{\mathfrak{p}} = (\det A)\, \eta^{(4)}(\vec{l}_+^{U_\alpha}, \vec{l}_-^{U_\alpha}, \vec{e}_1, \vec{e}_2)\Big|_{\mathfrak{p}} < 0,$$

which contradicts the fifth equation in (2.2.2) for U_β. Consequently $\{\vec{l}_+, \vec{l}_-\}$ does not depend on the chart, which proves the result. ■

Remark. From now on we will take the vector fields $\vec{l}_+$, $\vec{l}_-$ to be partially normalized to satisfy $l_{+\mu} l_-^\mu = -2$, as in the proof of the lemma. Note that these vector fields are then defined modulo a transformation $\vec{l}_+ \to F\vec{l}_+$, $\vec{l}_- \to \frac{1}{F}\vec{l}_-$, where F is a positive function on S. □

For a surface S, ∇^S will denote the covariant derivative associated with γ and $\vec{\Pi}$ and $\vec{H}$ will denote the second fundamental form vector and the mean curvature of S in M. The physical meaning of the causal character of $\vec{H}$ is closely related to the first variation of area, which we briefly discuss next. Let $\vec{\nu}$ be a normal variation vector on S, i.e. a vector defined in a neighbourhood of S in M which, on S, is orthogonal to S. Choose $\vec{\nu}$ to be compactly supported on S (which obviously places no restrictions when S itself is compact). The vector $\vec{\nu}$ generates a one-parameter local group $\{\varphi_\tau\}_{\tau\in I}$ of transformations where τ is the canonical parameter and $I \subset \mathbb{R}$ is an interval containing $\tau = 0$. We then define a one parameter family of surfaces $S_\tau \equiv \varphi_\tau(S)$, which obviously satisfies $S_{\tau=0} = S$. Let $|S_\tau|$ denote the area of the surface S_τ. The formula of the first variation of area states (see e.g. [36])

$$\delta_{\vec{\nu}}|S| \equiv \left.\frac{d|S_\tau|}{d\tau}\right|_{\tau=0} = \int_S H_\mu \nu^\mu \eta_S. \tag{2.2.3}$$

Remark. It is important to indicate that, when S is boundaryless, expression (2.2.3) holds regardless of whether the variation $\vec{\nu}$ is normal or not. This formula is valid for any dimensions of M and S, provided $\dim M > \dim S$. □

The first variation of area justifies the definition of a *minimal surface* as follows.

Definition 2.2.11 *A surface S is* **minimal** *if and only if $\vec{H} = 0$.*

According to (2.2.3), if $\vec{H}$ is timelike and future directed (resp. past directed) everywhere on S, then the area of S will decrease along any non-zero causal future (resp. past) direction. If a surface is such that its area does not increase for any future variation, one may say that the surface is, in some sense, trapped. Thus, according to the previous discussion, we find that the *trappedness* of a surface is intimately related with the causal character and time orientation of its mean curvature vector $\vec{H}$. In what follows, we will introduce various notions of trapped surface. For that, it will be useful to consider a null basis $\{\vec{l}_+, \vec{l}_-\}$ for the normal bundle of S in M, as before. Then, the mean curvature vector decomposes as

$$\vec{H} = -\frac{1}{2}\left(\theta^- \vec{l}_+ + \theta^+ \vec{l}_-\right), \tag{2.2.4}$$

where $\theta^+ \equiv {l_+}^\mu H_\mu$ and $\theta^- \equiv {l_-}^\mu H_\mu$ are the null expansions of S along $\vec{l}_+$ and $\vec{l}_-$, respectively. It is worth to remark that these null expansions $\theta^\pm$ are equal to the divergence on S of light rays (i.e. null geodesics) emerging orthogonally from S along $\vec{l}_\pm$. Thus, the negativity of both θ^+ and θ^- indicates the presence of strong gravitational fields which bend the light rays sufficiently so that both are contracting.

Thus, this leads to various concepts of trapped surfaces, as follows.

Definition 2.2.12 *A closed (i.e. compact and without boundary) surface is a:*

- **Trapped surface** *if $\theta^+ < 0$ and $\theta^- < 0$. Or equivalently, if $\vec{H}$ is timelike and future directed.*

- **Weakly trapped surface** *if $\theta^+ \leq 0$ and $\theta^- \leq 0$. Or equivalently, if $\vec{H}$ is causal and future directed.*

- **Marginally trapped surface** *if either, $\theta^+ = 0$ and $\theta^- \leq 0$ everywhere, or, $\theta^+ \leq 0$ and $\theta^- = 0$ everywhere. Equivalently, if $\vec{H}$ is future directed and either proportional to $\vec{l}_+$ or proportional to $\vec{l}_-$ everywhere.*

If the signs of the inequalities are reversed then we have trappedness along the past directed causal vectors orthogonal to S. Thus,

Definition 2.2.13 *A closed surface is a:*

- **Past trapped surface** *if $\theta^+ > 0$ and $\theta^- > 0$. Or equivalently if $\vec{H}$ is timelike and past directed.*

- **Past weakly trapped surface** *if $\theta^+ \geq 0$ and $\theta^- \geq 0$. Or equivalently if $\vec{H}$ is causal and past directed.*

- **Past marginally trapped surface** *if either, $\theta^+ = 0$ and $\theta^- \geq 0$ everywhere, or $\theta^+ \geq 0$ and $\theta^- = 0$ everywhere. Equivalently, $\vec{H}$ is past directed and either proportional to $\vec{l}_+$ or proportional to $\vec{l}_-$ everywhere.*

We also define "untrapped" surface as a kind of strong complementary of the above.

Definition 2.2.14 *A closed surface is* **untrapped** *if $\theta^+\theta^- < 0$, or equivalently if $\vec{H}$ is spacelike everywhere.*

Notice that, according to these definitions, a closed *minimal* surface is both weakly trapped and marginally trapped, as well as past weakly trapped and past marginally trapped.

Because of their physical meaning as indicators of strong gravitational fields, trapped surfaces are widely considered as good natural quasi-local replacements for black holes. Let us briefly recall the definition of a black hole which, as already mentioned in the Introduction, involves global hypotheses in the spacetime. First, it requires a proper definition of asymptotic flatness in terms of the conformal compactification of the spacetime (see e.g. Chapter 11 of [112]). Besides, it also requires that the spacetime is *strongly asymptotically predictable*, (see Chapter 12 of [112] for a precise definition). A strongly asymptotically predictable spacetime $(M, g^{(4)})$ is then said to contain a black hole if M is not contained in the causal past of future null infinity $J^-(\mathscr{I}^+)$. The **black hole region** $\mathcal{B}$ is defined as $\mathcal{B} = M \setminus J^-(\mathscr{I}^+)$. The topological boundary $\mathcal{H}_{\mathcal{B}}$ of $\mathcal{B}$ in M is called the **event horizon**. Similarly, we can define the **white hole region** $\mathcal{W}$ as the complementary of the causal future of past null infinity, i.e. $M \setminus J^+(\mathscr{I}^-)$, and the **white hole event horizon** $\mathcal{H}_{\mathcal{W}}$ as its topological boundary. Finally, the **domain of outer communications** is defined as $M_{DOC} \equiv J^-(\mathscr{I}^+) \cap J^+(\mathscr{I}^-)$. Hawking and Ellis show (see Chapter 9.2 in [65]) that weakly trapped surfaces lie inside the black hole region in a spacetime provided this spacetime is future asymptotically predictable. However, as we already pointed out in the Introduction, the study of trapped surfaces is specially interesting when no global assumptions are imposed on the spacetime and the concept of black hole is not available. It is worth to remark that trapped surfaces are also fundamental ingredients in several versions of singularity theorems of General Relativity (see e.g. Chapter 9 in [112]).

Note that all the surfaces introduced above are defined by restricting both null expansions θ^+ and θ^-. When only one of the null expansions is restricted, other interesting types of surfaces are obtained: the *outer* trapped surfaces, which will be the fundamental objects of this thesis.

Again, consider a surface S. Suppose that for some reason one of the future null directions can be geometrically selected so that it points into the "outer" direction of S (shortly, we will find a specific setting where this selection is meaningful). In that situation we will always denote by $\vec{l}_+$ the vector pointing along this outer null direction. We will say that $\vec{l}_+$ is the future outer null direction, and similarly, $\vec{l}_-$ will be the future inner null direction. We define the following types of surfaces (c.f. Figure 2.1).

Definition 2.2.15 *A closed surface is:*

- **Outer trapped** *if* $\theta^+ < 0$.
- **Weakly outer trapped** *if* $\theta^+ \leq 0$.
- **Marginally outer trapped (MOTS)** *if* $\theta^+ = 0$.
- **Outer untrapped** *if* $\theta^+ > 0$.

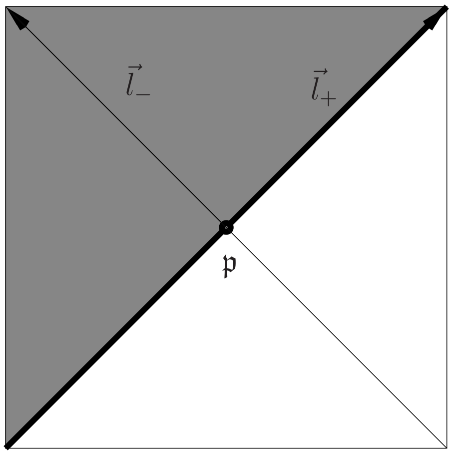


Figure 2.1: This figure represents the normal space to S in M at a point $\mathfrak{p} \in S$. If S is outer trapped, the mean curvature vector $\vec{H}$ points into the shaded region. If S is a MOTS, $\vec{H}$ points into the direction of the bold line.

As before, these definitions depend on the time orientation of the spacetime. If the time orientation is reversed but the notion of *outer* is unambiguous, then $-\vec{l}_-$ becomes the new future outer null direction. Since the null expansion of $-\vec{l}_-$ is $-\theta^-$, the following definitions become natural (c.f. Figure 2.2).

Definition 2.2.16 *A closed surface is:*

- **Past outer trapped** *if* $\theta^- > 0$.
- **Past weakly outer trapped** *if* $\theta^- \geq 0$.
- **Past marginally outer trapped (past MOTS)** *if* $\theta^- = 0$.
- **Past outer untrapped** *if* $\theta^- < 0$.

As for weakly trapped surfaces, weakly outer trapped surfaces are always inside the black hole region provided the spacetime is strongly asymptotically predictable. In fact, in one of the simplest dynamical situations, namely the Vaidya

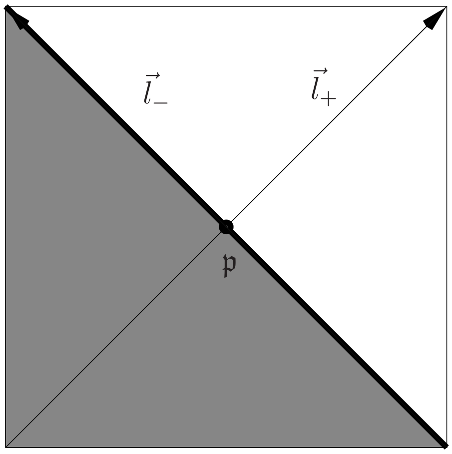


Figure 2.2: On the normal space $N_{\mathfrak{p}}S$ for any point $\mathfrak{p} \in S$, the mean curvature vector $\vec{H}$ points into the shaded region if S is past outer trapped, and into the direction of the bold line if S is a past MOTS.

spacetime, Ben-Dov has proved [14] that the event horizon is the boundary of the spacetime region containing weakly outer trapped surfaces, proving in this particular case a previous conjecture by Eardley [52]. On the other hand, Bengtsson and Senovilla have shown [15] that the spacetime region containing weakly trapped surfaces does not extend to the event horizon. This result suggests that the concept of weakly outer trapped surface does capture the essence of a black hole better than that of weakly trapped surface.

Two other interesting classes of surfaces that also depend on a choice of outer direction are the so-called *generalized trapped surfaces* and its marginal case, *generalized apparent horizons*. They were specifically introduced by Bray and Khuri while studying a new approach to prove the Penrose inequality [20].

Definition 2.2.17 *A closed surface is a:*

- **Generalized trapped surface** *if* $\theta^+|_{\mathfrak{p}} \leq 0$ *or* $\theta^-|_{\mathfrak{p}} \geq 0$ *at each point* $\mathfrak{p} \in S$*.*

- **Generalized apparent horizon** *if either* $\theta^+|_{\mathfrak{p}} = 0$ *with* $\theta^-|_{\mathfrak{p}} \leq 0$ *or* $\theta^-|_{\mathfrak{p}} = 0$ *with* $\theta^+|_{\mathfrak{p}} \geq 0$ *at each point* $\mathfrak{p} \in S$*.*

It is clear from Figures 2.1, 2.2 and 2.3 that the set of generalized trapped surfaces includes both the set of weakly outer trapped surfaces and the set of past weakly outer trapped surfaces as particular cases.

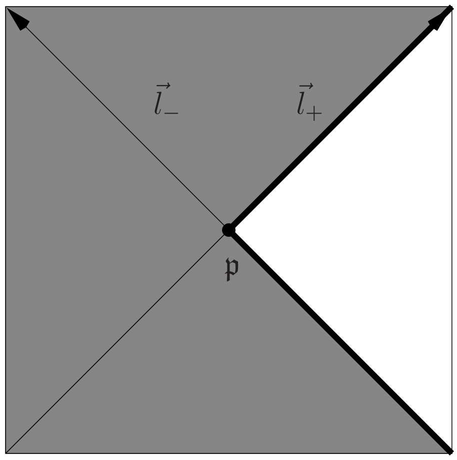


Figure 2.3: This figure represents the normal space of a surface S in M at a point $\mathfrak{p} \in S$. For generalized trapped surfaces, the mean curvature vector $\vec{H}$ points into the shaded region. For generalized apparent horizons, $\vec{H}$ points into the direction of the bold line.

In this thesis we will often consider surfaces embedded in a spacelike hypersurface $\Sigma \subset M$. For this reason, it will be useful to give a (3+1) decomposition of the null expansions and to reformulate the previous definitions in terms of objects defined directly on Σ.

Definition 2.2.18 *A* **hypersurface** Σ *of* M *is an embedded, connected spacelike submanifold, possibly with boundary, of codimension 1.*

Let us consider a hypersurface Σ of M and denote by g its induced metric, by $\vec{K}$ its second fundamental form vector and by K the second fundamental form, defined as $K(\vec{X},\vec{Y}) = -\mathbf{n}(\vec{K}(\vec{X},\vec{Y}))$, where $\mathbf{n}$ is the unit, future directed, normal 1-form to Σ and $\vec{X},\vec{Y} \in \mathfrak{X}(\Sigma)$.

Consider a surface S embedded in (Σ, g, K) As before, we denote by γ, $\vec{\Pi}$ and $\vec{H}$ the induced metric, the second fundamental form vector and the mean curvature vector of S as a submanifold of $(M, g^{(4)})$, respectively. As a submanifold of Σ, S will also have a second fundamental form vector $\vec{\kappa}$ and a mean curvature vector $\vec{p}$. From their definitions, we immediately have

$$\vec{\Pi}(\vec{X},\vec{Y}) = \vec{K}(\vec{X},\vec{Y}) + \vec{\kappa}(\vec{X},\vec{Y}),$$

where $\vec{X},\vec{Y} \in \mathfrak{X}(S)$. Taking trace on S we find

$$\vec{H} = \vec{p} + \gamma^{AB}\vec{K}_{AB}, \tag{2.2.5}$$

where $\vec{K}_{AB}$ is the pull-back of $\vec{K}_{ij}$ $(i, j = 1, 2, 3)$ onto S. Assume that an outer null direction $\vec{l}_+$ can be selected on S. Then, after a suitable rescaling of $\vec{l}_+$ and $\vec{l}_-$, we can define $\vec{m}$ univocally on S as the unit vector tangent to Σ which satisfies

$$\vec{l}_+ = \vec{n} + \vec{m}, \tag{2.2.6}$$
$$\vec{l}_- = \vec{n} - \vec{m}. \tag{2.2.7}$$

By construction, $\vec{m}$ is normal to S in Σ and will be denoted as the *outer normal.*

Multiplying (2.2.5) by $\vec{l}_+$ and by $\vec{l}_-$ we find

$$\theta^{\pm} = \pm p + q, \tag{2.2.8}$$

where $p \equiv p_i m^i$ and $q \equiv \gamma^{AB} K_{AB}$. All objects in (2.2.8) are intrinsic to Σ. This allows us to reformulate the definitions above in terms of p and q. The following table summarizes the types of surfaces mostly used in this thesis.

Outer trapped surface	$p < -q$
Weakly outer trapped surface	$p \leq -q$
Marginally outer trapped surface (MOTS)	$p = -q$
Outer untrapped surface	$p > -q$
Past outer trapped surface	$p < q$
Past weakly outer trapped surface	$p \leq q$
Past marginally outer trapped surface (past MOTS)	$p = q$
Past outer untrapped surface	$p > q$
Generalized trapped surface	$p \leq \lvert q \rvert$
Generalized apparent horizon	$p = \lvert q \rvert$

Table I: Definitions of various types of trapped surfaces in terms of the mean curvature p of $S \subset \Sigma$ and the trace q on S of the second fundamental form of Σ in M.

Having defined the main types of surfaces used in this thesis, let us next consider the important concept of stability of a MOTS.

2.2.2 Stability of marginally outer trapped surfaces (MOTS)

Let us first recall the concept of stability for minimal surfaces. Let S be a closed minimal surface embedded in a Riemannian 3-dimensional manifold (Σ, g). From

(2.2.3), S is an extremal of area for all variations (normal or not). In order to study whether this extremum is a minimum, a maximum or a saddle point, it is necessary to analyze the second variation of area. A minimal surface is called *stable* if the second variation of area is non-negative for all smooth variations. This definition becomes operative once an explicit form for the second variation is obtained. For closed minimal surfaces the crucial object is the so-called *stability operator*, defined as follows. Consider a variation vector $\psi\vec{m}$ normal to S within Σ. Let us denote by a sub-index τ the magnitudes which correspond to the surfaces $S_\tau = \varphi_\tau(S)$ (where, as before, $\{\varphi_\tau\}_{\tau\in I\subset\mathbb{R}}$ denotes the one-parameter local group of transformations generated by any vector $\vec{\nu}$ satisfying $\vec{\nu}|_S = \psi\vec{m}$). For any covariant tensor Γ defined on S, let us define the variation of Γ along $\psi\vec{m}$ as $\delta_{\psi\vec{m}}\Gamma \equiv \frac{d}{d\tau}\left[\varphi_\tau^*(\Gamma_\tau)\right]\big|_{\tau=0}$, where φ^*_τ denotes the pull-back of φ_τ (this definition does not depend on the extension of the vector $\psi\vec{m}$ outside S). The stability operator $L^{min}_{\vec{m}}$ is then defined as

$$L^{min}_{\vec{m}}\psi \equiv \delta_{\psi\vec{m}}p = -\Delta_S\psi - (R^{\Sigma}{}_{ij}m^i m^j + \kappa_{ij}\kappa^{ij})\psi, \qquad (2.2.9)$$

where $\Delta_S = \nabla^S_A\nabla^{SA}$ is the Laplacian on S and $R^{\Sigma}{}_{ij}$ denotes the Ricci tensor of (Σ, g). The second equality follows from a direct computation (see e.g. [36]).

In terms of the stability operator, the formula for the second variation of area of a closed minimal surface is given by

$$\delta^2_{\psi\vec{m}}|S| = \int_S \psi L^{min}_{\vec{m}}\psi\eta_S.$$

The operator $L^{min}_{\vec{m}}$ is linear, elliptic and formally self-adjoint (see Appendix B for the definitions). Being self-adjoint implies that the principal eigenvalue ϱ can be represented by the Rayleigh-Ritz formula (B.2), and therefore the second variation of area can be bounded according to

$$\delta^2_{\psi\vec{m}}|S| \geq \varrho \int_S \psi^2\eta_S,$$

where equality holds when ψ is a principal eigenfunction (i.e. an eigenfunction corresponding to ϱ). This implies that $\delta^2_{\psi\vec{m}}|S| \geq 0$ for all smooth variations is equivalent to $\varrho \geq 0$. Thus, *a minimal surface is stable if and only if* $\varrho \geq 0$.

A related construction can be performed for MOTS. Consider a MOTS S embedded in a spacelike hypersurface Σ of a spacetime M. As embedded submanifolds of Σ, MOTS are not minimal surfaces in general. Consequently, any connection between stability and the second variation of area is lost. However, the stability for minimal surfaces involves the sign of the variation $\delta_{\psi\vec{m}}p$ (see

(2.2.9)), so it is appropriate to define stability of MOTS in terms of the sign of first variations of θ^+.

A formula for the first variation of θ^+ was derived by Newman in [92] for arbitrary immersed spacelike submanifolds. The derivation was simplified by Andersson, Mars and Simon in [3].

Lemma 2.2.19 *Consider a surface S embedded in a spacetime $(M, g^{(4)})$. Let $\{\vec{l}_+, \vec{l}_-\}$ be a future directed null basis in the normal bundle of S in M, partially normalized to satisfy $l_{+\mu} l_-^\mu = -2$. Any variation vector $\vec{\nu}$ can be decomposed on S as $\vec{\nu} = \vec{\nu}^\parallel + b\vec{l}_+ - \frac{u}{2}\vec{l}_-$, where $\vec{\nu}^\parallel$ is tangent to S and b and u are functions on S. Then,*

$$\begin{aligned} \delta_{\vec{\nu}}\theta^+ &= -\frac{\theta^+}{2} l_-{}^\mu \delta_{\vec{\nu}} l_{+\mu} + \vec{\nu}^\parallel(\theta^+) - b\left(\Pi^\mu{}_{AB}\Pi^{\nu AB} l_{+\mu} l_{+\nu} + G_{\mu\nu} l_+{}^\mu l_+{}^\nu\right) - \Delta_S u \\ &\quad + 2s^A \nabla^S_A u + \frac{u}{2}\left(R_S - H^2 - G_{\mu\nu} l_+{}^\mu l_-{}^\nu - 2s_A s^A + 2\nabla^S_A s^A\right), \end{aligned} \quad (2.2.10)$$

where R_S denotes the scalar curvature of S, $H^2 = H_\mu H^\mu$ and $s_A = -\frac{1}{2} l_{-\mu} \nabla_{\vec{e}_A} l_+{}^\mu$, with $\{\vec{e}_A\}$ being a local basis for TS.

Expression (2.2.10) can be particularized when the variation is restricted to Σ, i.e. when $\vec{\nu} = \psi\vec{m}$ for an arbitrary function ψ. Writing $\vec{l}_\pm = \vec{n} \pm \vec{m}$ as before, we have $\vec{\nu} = \frac{\psi}{2}(\vec{l}_+ - \vec{l}_-)$ and hence $\vec{\nu}^\parallel = 0$, $b = \frac{\psi}{2}$, $u = \psi$. As a consequence of Lemma 2.2.19 we have the following [3].

Definition 2.2.20 *The* **stability operator** $L_{\vec{m}}$ *for a MOTS S is defined by*

$$L_{\vec{m}}\psi \equiv \delta_{\psi\vec{m}}\theta^+ = -\Delta_S\psi + 2s^A\nabla^S_A\psi + \left(\frac{1}{2}R_S - Y - s_A s^A + \nabla^S_A s^A\right)\psi, \quad (2.2.11)$$

where

$$Y \equiv \frac{1}{2}\Pi^\mu_{AB}\Pi^{\nu AB} l_{+\mu} l_{+\nu} + G_{\mu\nu} l^\mu_+ n^\nu. \quad (2.2.12)$$

Remark. In terms of objects on Σ, a simple computation using $\vec{l}_\pm = \vec{n} \pm \vec{m}$ shows that $s_A = m^i e^j_A K_{ij}$. □

If we consider a variation along $\vec{l}_+$, then (2.2.10) implies that, on a MOTS,

$$\delta_{\psi\vec{l}_+}\theta^+ = -\psi W, \quad (2.2.13)$$

where

$$W = \Pi^\mu_{AB}\Pi^{\nu AB} l_{+\mu} l_{+\nu} + G_{\mu\nu} l^\mu_+ l^\nu_+. \quad (2.2.14)$$

This is the well-known Raychaudhuri equation for a MOTS (see e.g. [112]).

Note that W is non-negative provided the NEC holds and Y is non-negative if the DEC holds (recall that $\vec{n}$ is timelike).

The operator $L_{\vec{m}}$ is linear and elliptic which implies that it has a discrete spectrum. However, due to the presence of a first order term, it is not formally self-adjoint (see Appendix B) in general. Nevertheless, it is still true (c.f. Lemma (B.5 in Appendix B)) that there exists an eigenvalue ϱ with smallest real part. This eigenvalue is called the *principal eigenvalue* and it has the following properties:

1. It is real.

2. Its eigenspace (the set of smooth real functions ψ on S satisfying $L_{\vec{m}}\psi = \varrho\psi$) is one-dimensional.

3. An eigenfunction ψ of ϱ vanishes at one point $\mathfrak{p} \in S$ if and only if it vanishes everywhere on S (i.e. the principal eigenfunctions do not change sign).

The stability of minimal surfaces could be rewritten in terms of the sign of the principal eigenvalue of its stability operator. In [2], [3] the following definition of stability of MOTS is put forward.

Definition 2.2.21 *A MOTS $S \subset \Sigma$ is* **stable in** Σ *if the principal eigenvalue ϱ of the stability operator $L_{\vec{m}}$ is non-negative. S is* **strictly stable in** Σ *if $\varrho > 0$.*

For simplicity, since no confusion will arise, we will refer to *stability in* Σ simply as *stability*.

For stable MOTS, there is no scalar quantity which is non-decreasing for arbitrary variations, like the area for stable minimal surfaces. However, in the minimal surface case, the formula

$$< \phi, \psi >_{L^2} \varrho =< L^{min}_{\vec{m}}\phi, \psi >_{L^2} =< \phi, L^{min}_{\vec{m}}\psi >_{L^2},$$

where ϕ is a principal eigenfunction of $L^{min}_{\vec{m}}$, implies that if there exists a positive variation $\psi\vec{m}$ for which $\delta_{\psi\vec{m}}p \geq 0$, then $\varrho \geq 0$ and the minimal surface is stable. A similar result can be proven for MOTS [3]:

Proposition 2.2.22 *Let $S \subset \Sigma$ be a MOTS. Then S is stable if and only if there exists a function $\psi \geq 0$, $\psi \not\equiv 0$ on S such that $\delta_{\psi\vec{m}}\theta^+ \geq 0$. Furthermore, S is strictly stable if and only if, in addition, $\delta_{\psi\vec{m}}\theta^+ \not\equiv 0$.*

Remark. For the case of *past* MOTS simply change $\vec{n} \to -\vec{n}$, $\vec{l}_+ \to -\vec{l}_-$, $\vec{l}_- \to -\vec{l}_+$, $s_A \to -s_A$ and $\theta^+ \to -\theta^-$ in equations (2.2.11), (2.2.12), (2.2.13), (2.2.14) and, also, in Proposition 2.2.22. □

Thus, Proposition 2.2.22 tells us that a (resp. past) MOTS S is strictly stable if and only if there exists an outer variation with strictly increasing (resp. decreasing) θ^+ (resp. θ^-). This suggests that the presence of surfaces with negative θ^+ (resp. positive θ^-) outside S may be related with the stability property of S. This can be made precise by introducing the following notion.

Definition 2.2.23 *A (resp. past) MOTS $S \subset \Sigma$ is* **locally outermost** *if there exists a two-sided neighbourhood of S on Σ whose exterior part does not contain any (resp. past) weakly outer trapped surface.*

The following proposition gives the relation between these concepts [2].

Proposition 2.2.24

1. *A strictly stable MOTS (or past MOTS) is necessarily locally outermost.*
2. *A locally outermost MOTS (or past MOTS) is necessarily stable.*
3. *None of the converses is true in general.*

2.2.3 The trapped region

In this section we will extend the notion of locally outermost to a *global* concept and state a theorem by Andersson and Metzger [4] on the existence, uniqueness and regularity of the outermost MOTS on a spacelike hypersurface Σ. We will also see that an analogous result holds for the outermost generalized apparent horizon (Eichmair, [53]). Both results will play a fundamental role throughout this thesis.

The result by Andersson and Metzger is local in the sense that it works for any *compact* spacelike hypersurface Σ with boundary $\partial\Sigma$ as long as the boundary $\partial\Sigma$ splits in two disjoint non-empty components $\partial\Sigma = \partial^-\Sigma \cup \partial^+\Sigma$. Neither of these components is assumed to be connected a priori. Andersson and Metzger deal with surfaces which are *bounding with respect to* the boundary $\partial^+\Sigma$ which plays the role of outer untrapped *barrier*. Both concepts are defined as follows.

Definition 2.2.25 *Consider a spacelike hypersurface Σ possibly with boundary. A closed surface $S_b \subset \Sigma$ is a* **barrier with interior** Ω_b *if there exists a manifold with boundary Ω_b which is topologically closed and such that $\partial\Omega_b = S_b \bigcup \underset{a}{\cup}(\partial\Sigma)_a$, where $\underset{a}{\cup}(\partial\Sigma)_a$ is a union (possibly empty) of connected components of $\partial\Sigma$.*

Remark. For simplicity, when no confusion arises, we will often refer to a barrier S_b with interior Ω_b simply as a *barrier S_b*. □

The concept of a barrier will give us a criterion to define the exterior and the interior of a special type of surfaces called *bounding*. More precisely,

Definition 2.2.26 *Consider a spacelike hypersurface Σ possibly with boundary with a barrier S_b with interior Ω_b. A surface $S \subset \Omega_b \setminus S_b$ is* **bounding with respect to the barrier** S_b *if there exists a compact manifold $\Omega \subset \Omega_b$ with boundary such that $\partial\Omega = S \cup S_b$. The set $\Omega \setminus S$ will be called the* **exterior** *of S in Ω_b and $(\Omega_b \setminus \Omega) \cup S$ the* **interior** *of S in Ω_b.*

Remark. Note that a surface S which is bounding with respect to a barrier S_b is always disjoint to S_b and that its exterior is always not empty. Again, for simplicity and when no confusion arises, we will often refer to a surface which is bounding with respect a barrier simply as a *bounding surface*. Notice that, in the topology of Ω_b, the exterior of a bounding surface S in Ω_b is topologically open (because for every point $\mathfrak{p} \in \partial\Omega_b$ there exists an open set $U \subset \Omega_b$ such that $\mathfrak{p} \in U$), while its interior is topologically closed. For graphic examples of surfaces which are bounding with respect to a barrier see figures 2.4 and 2.5. □

The concept of bounding surface allows for a meaningful definition of *outer null direction*. For that, define the vector $\vec{m}$ as the unit vector normal to S in Σ which points into the exterior of S in Ω_b. For S_b, $\vec{m}$ will be taken to point outside of Ω_b. Then, we will select the outer and the inner null vectors, $\vec{l}_+$ and $\vec{l}_-$ as those null vectors orthogonal to S or S_b which satisfy equations (2.2.6) and (2.2.7), respectively.

Definition 2.2.27 *Given two surfaces S_1 and S_2 which are bounding with respect to a barrier S_b, we will say that S_1 encloses S_2 if the exterior of S_2 contains the exterior of S_1.*

Definition 2.2.28 *A (past) MOTS $S \subset \Sigma$ which is bounding with respect to a barrier S_b is* **outermost** *if there is no other (past) weakly outer trapped surface in Σ which is bounding with respect to S_b and enclosing S.*

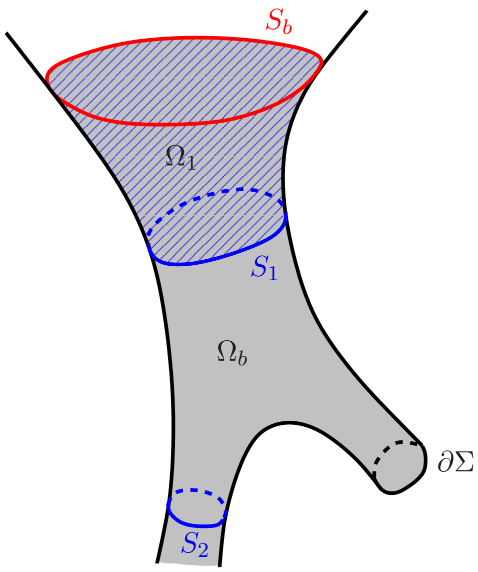


Figure 2.4: In this graphic example, the surface S_b (in red) is a barrier with interior Ω_b (in grey). The surface S_1 is bounding with respect to S_b with Ω_1 (the stripped area) being its exterior in Ω_b. The surface S_2 fails to be bounding with respect to S_b because its "exterior" would contain $\partial\Sigma$.

Since bounding surfaces split Ω_b into an exterior and an interior region, it is natural to consider the points inside a bounding weakly outer trapped surface S as "trapped points". The region containing trapped points is called *weakly outer trapped region* and will be essential for the formulation of the result by Andersson and Metzger. More precisely,

Definition 2.2.29 *Consider a spacelike hypersurface containing a barrier S_b with interior Ω_b. The* **weakly outer trapped region** *T^+ of Ω_b is the union of the interiors of all bounding weakly outer trapped surfaces in Ω_b.*

Analogously,

Definition 2.2.30 *The* **past weakly outer trapped region** *T^- of Ω_b is the union of the interiors of all bounding past weakly outer trapped surfaces in Ω_b.*

The fundamental result by Andersson and Metzger, which will be an important tool in this thesis, reads as follows.

Theorem 2.2.31 (Andersson, Metzger, 2009 [4]) *Consider a compact spacelike hypersurface $\tilde{\Sigma}$ with boundary $\partial\tilde{\Sigma}$. Assume that the boundary can be*

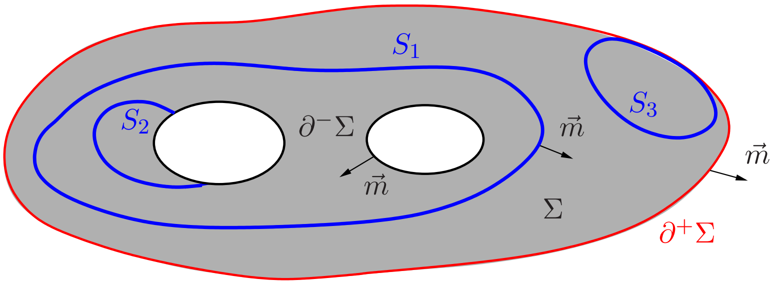


Figure 2.5: A manifold Σ with boundary $\partial\Sigma = \partial^-\Sigma \cup \partial^+\Sigma$. The boundary $\partial^+\Sigma$ is a barrier whose interior coincides with Σ. The surface S_1 is bounding with respect to $\partial^+\Sigma$, while S_2 and S_3 fail to be bounding. The figure also shows the outer normal $\vec{m}$ as defined in the text.

split in two non-empty disjoint components $\partial\tilde{\Sigma} = \partial^-\tilde{\Sigma} \cup \partial^+\tilde{\Sigma}$ (neither of which are necessarily connected) and take $\partial^+\tilde{\Sigma}$ as a barrier with interior $\tilde{\Sigma}$. Suppose that $\theta^+[\partial^-\tilde{\Sigma}] \leq 0$ and $\theta^+[\partial^+\tilde{\Sigma}] > 0$ (with respect to the outer normals defined above). Then the topological boundary $\partial^{top}T^+$ of the weakly outer trapped region of $\tilde{\Sigma}$ is a smooth MOTS which is bounding with respect to $\partial^+\tilde{\Sigma}$ and stable.

Remark. Since no bounding MOTS can penetrate into the exterior of $\partial^{top}T^+$, by definition, this theorem shows the existence, uniqueness and smoothness of the outermost bounding MOTS in a compact hypersurface. Note also that another consequence of this result is the fact that the set T^+ is topologically closed (because it is the interior of the bounding surface $\partial^{top}T^+$). □

The proof of this theorem uses the Gauss-Bonnet Theorem in several places and, therefore, this result is valid only in (3+1) dimensions.

If we reverse the time orientation of the spacetime, an analogous result for the topological boundary of the past weakly outer trapped region T^- follows. Indeed, if the hypotheses on the sign of the outer null expansion of the components of $\partial\tilde{\Sigma}$ are replaced by $\theta^-[\partial^-\tilde{\Sigma}] \geq 0$ and $\theta^-[\partial^+\tilde{\Sigma}] < 0$ then the conclusion is that $\partial^{top}T^-$ is a smooth past MOTS which is bounding with respect to $\partial^+\tilde{\Sigma}$ and stable.

As we mentioned before, a similar result for the existence of the outermost generalized apparent horizon also exists. It has been recently obtained by Eichmair [53].

Theorem 2.2.32 (Eichmair, 2009 [53]) *Let $(\tilde{\Sigma}, g, K)$ be a compact n-dimensional spacelike hypersurface in an (n+1)-dimensional spacetime, with* $3 \leq$

$n \leq 7$ and boundary $\partial\tilde{\Sigma}$. Assume that the boundary can be split in two non-empty disjoint components $\partial\tilde{\Sigma} = \partial^-\tilde{\Sigma} \cup \partial^+\tilde{\Sigma}$ (neither of which are necessarily connected) and take $\partial^+\tilde{\Sigma}$ as a barrier with interior $\tilde{\Sigma}$. Suppose that the inner boundary $\partial^-\tilde{\Sigma}$ is a generalized trapped surface, and the outer boundary satisfies $p > |q|$ with respect to the outer normals defined above.

Then there exists a unique $C^{2,\alpha}$ (i.e. belonging to the Hölder space $C^{2,\alpha}$, with $0 < \alpha \leq 1$, see Appendix B) generalized apparent horizon S which is bounding with respect to $\partial^+\tilde{\Sigma}$ and outermost (i.e. there is no other bounding generalized trapped surface in $\tilde{\Sigma}$ enclosing S). Moreover, S has smaller area than any other surface enclosing it.

The proof of this result does not use the Gauss-Bonnet theorem or any other specific property of 3-dimensional spaces, so it not restricted to (3+1) dimensions. However, it is based on regularity of minimal surfaces, which implies that the dimension of $\tilde{\Sigma}$ must be at most seven (in higher dimensions minimal hypersurfaces need not be regular everywhere, see e.g. [59]).

The area minimizing property of the outermost bounding generalized apparent horizon makes this type of surfaces potentially interesting for the Penrose inequality, as we will discuss in the next section.

2.3 The Penrose inequality

The Penrose inequality involves the concept of the total ADM mass of a spacetime, so we start with a brief discussion about mass in General Relativity.

The notion of *energy* in General Relativity is not as clear as in other physical theories. The energy-momentum tensor $T_{\mu\nu}$ represents the matter contents of a spacetime and therefore should contribute to the total energy of a spacetime. However, the *gravitational field*, represented by the metric tensor $g^{(4)}$, must also contribute to the total energy of the spacetime. In agreement with the Newtonian limit, a suitable *gravitational energy density* should be an expression quadratic in the first derivatives of the metric $g^{(4)}$. However, since at any point we can make the metric to be Minkowskian and the Christoffel symbols to vanish, there is no non-trivial scalar object constructed from the metric and its first derivatives alone. Therefore, a natural notion of energy density in General Relativity does not exist. The same problem is also found in other geometric theories of gravity. Nevertheless, there does exist a useful notion of the *total energy* in the so-called asymptotically flat spacetimes.

The term *asymptotic flatness* was introduced in General Relativity to express the idea of a spacetime corresponding to an isolated system. It involves restrictions on the spacetime “far away” form the sources. There are several notions of asymptotic flatness according to the type of infinity considered (see e.g. Chapter 11.1 of [112]), namely limits along null directions (null infinity) or limits along spacelike directions (spacelike infinity). The idea is to define the mass as integrals in the asymptotic region where the gravitational field is sufficiently weak so that integrals become meaningful (i.e. independent of the coordinate system). According to the type of infinity considered there are two different concepts: the *Bondi energy-momentum* where the integral is taken at null infinity and the *ADM energy-momentum* where the integral is taken at spatial infinity. Both are vectors in a suitable four dimensional vector space and transform as a Lorentz vector under suitable transformations. Moreover, the Lorentz length of this vector is either a conserved quantity upon evolution (ADM) or monotonically decreasing in advanced time (Bondi). An interesting and more precise discussion about the definitions of both Bondi and ADM energy-momentum tensors can be found in Chapter 11.2 of [112]. Because of its relation with the Penrose inequality we are specially interested in the ADM energy-momentum. To make these concepts precise we need to define first *asymptotic flatness* for spacelike hypersurfaces.

Definition 2.3.1 *An* **asymptotically flat end** *of a spacelike hypersurface (Σ, g, K) is a subset $\Sigma_0^\infty \subset \Sigma$ which is diffeomorphic to $\mathbb{R}^3 \setminus \overline{B_R}$, where B_R is an open ball of radius R. Moreover, in the Cartesian coordinates $\{x^i\}$ induced by the diffeomorphism, the following decay holds*

$$g_{ij} - \delta_{ij} = O^{(2)}(1/r), \qquad K_{ij} = O^{(2)}(1/r^2), \tag{2.3.1}$$

where $r = |x| = \sqrt{x^i x^j \delta_{ij}}$.

Here, a function $f(x^i)$ is said to be $O^{(k)}(r^n), k \in \mathbb{N} \cup \{0\}$ if $f(x^i) = O(r^n)$, $\partial_j f(x^i) = O(r^{n-1})$ and so on for all derivatives up to and including the k-th ones.

Definition 2.3.2 *A spacelike hypersurface (Σ, g, K), possibly with boundary, is* **asymptotically flat** *if $\Sigma = \mathcal{K} \cup \Sigma^\infty$, where $\mathcal{K}$ is a compact set and $\Sigma^\infty = \underset{a}{\cup} \Sigma_a^\infty$ is a finite union of asymptotically flat ends Σ_a^∞.*

Definition 2.3.3 *Consider a spacelike hypersurface (Σ, g, K) with a selected asymptotically flat end Σ_0^∞. Then, the ADM energy-momentum $\mathbf{P}_{ADM}$ associated*

with Σ_0^∞ is defined as the spacetime vector with components

$$P_{ADM0} = E_{ADM} \equiv \lim_{r\to\infty} \frac{1}{16\pi} \sum_{j=1}^{3} \int_{S_r} (\partial_j g_{ij} - \partial_i g_{jj})\, dS^i, \tag{2.3.2}$$

$$P_{ADMi} = p_{ADMi} \equiv \lim_{r\to\infty} \frac{1}{8\pi} \int_{S_r} (K_{ij} - g_{ij} trK)\, dS^j, \tag{2.3.3}$$

where $\{x^i\}$ are the Cartesian coordinates induced by the diffeomorphism which defines the asymptotically flat end, S_r is the surface at constant r and $dS^i = m^i dS$ with $\vec{m}$ being the outward unit normal and dS the area element.

The quantity E_{ADM} is called the ADM energy while $\mathbf{p}_{ADM}$ the ADM spatial momentum.

Definition 2.3.4 *The ADM mass is defined as*

$$M_{ADM} = \sqrt{E_{ADM}^2 - \delta^{ij} P_{ADMi} P_{ADMj}}.$$

A priori, these definitions depend on the choice of the coordinates $\{x^i\}$. However, the decay in g and K at infinity implies that $\mathbf{P}_{ADM}$ is indeed a geometric quantity provided $G^{(4)}_{\mu\nu} n^\mu$ decays as $1/r^4$ at infinity [5]. The notion of ADM mass is in fact independent of the coordinates as long as the decay (2.3.1) is replaced by

$$g_{ij} - \delta_{ij} = O^{(2)}(1/r^\alpha), \qquad K_{ij} = O^{(1)}(1/r^{1+\alpha}), \tag{2.3.4}$$

with $\alpha > \frac{1}{2}$ [9].

A fundamental property of the ADM energy-momentum is its causal character. The Positive Mass Theorem (PMT) of Schoen and Yau [104] (also proven by Witten [114] using spinors) establishes that the ADM energy is non-negative and the ADM mass is real (c.f. Section 8.2 of [110] for further details). More precisely,

Theorem 2.3.5 (Positive mass theorem (PMT), Schoen, Yau, 1981) *Consider an asymptotically flat spacelike hypersurface (Σ, g, K) without boundary satisfying the DEC. Then the total ADM energy-momentum $\vec{P}_{ADM}$ is a future directed causal vector. Furthermore, $\vec{P}_{ADM} = 0$ if and only if (Σ, g, K) is a slice of the Minkowski spacetime.*

The global conditions required for the PMT were relaxed in [10] where Σ was allowed to be complete and contain an asymptotically flat end instead of being necessarily asymptotically flat (see Theorem 2.4.12 below). The PMT has also been extended to other situations of interest. Firstly, it holds for spacelike

hypersurfaces admitting corners on a surface, provided the mean curvatures of the surface from one side and the other satisfy the right inequality [87]. It has also been proved for spacelike hypersurfaces *with boundary* provided this boundary is composed by either future or past weakly outer trapped surfaces [57]. Since future weakly outer trapped surfaces are intimately related with the existence of black holes (as we have already pointed out above), this type of PMT is usually referred to as *PMT for black holes*. Having introduced these notions we can now describe the Penrose inequality.

During the seventies, Penrose [97] conjectured that the total ADM mass of a spacetime containing a black hole that settles down to a stationary state must satisfy the inequality

$$M_{ADM} \geq \sqrt{\frac{|\mathscr{H}|}{16\pi}}, \tag{2.3.5}$$

where $|\mathscr{H}|$ is the area of the event horizon at one instant of time. Moreover, equality happens if and only if the spacetime is the Schwarzschild spacetime. The plausibility argument by Penrose goes as follows [97]. Assume a spacetime $(M, g^{(4)})$ which is globally well-behaved in the sense of being strongly asymptotically predictable and admitting a complete future null infinity $\mathscr{I}^+$ (see [112] for definitions). Suppose that M contains a non-empty black hole region. The black hole event horizon $\mathcal{H}_{\mathcal{B}}$ is a null hypersurface at least Lipschitz continuous. Next, consider a spacelike Cauchy hypersurface $\Sigma \subset M$ (see e.g. Chapter 8 of [112] for the definition of a Cauchy hypersurface) with ADM mass M_{ADM}. Clearly $\mathcal{H}_{\mathcal{B}}$ and Σ intersect in a two-dimensional Lipschitz manifold. This represents the event horizon at one instant of time. Let us denote by $\mathscr{H}$ this intersection and by $|\mathscr{H}|$ its area (the manifold is almost everywhere C^1 so the area makes sense). Consider now any other cut $\mathscr{H}_1$ lying in the causal future of $\mathscr{H}$. The black hole area theorem [63], [64], [43] states that $|\mathscr{H}_1| \geq |\mathscr{H}|$ provided the NEC holds. Physically, it is reasonable to expect that the spacetime settles down to some vacuum equilibrium configuration (if an electromagnetic field is present, the conclusions would be essentially the same). Then, the uniqueness theorems for stationary black holes (which hold under suitable assumptions [42], [49]) imply that the spacetime must approach the Kerr spacetime. In the Kerr spacetime the area of any cut of the event horizon $\mathscr{H}_{Kerr}$ takes the value $|\mathscr{H}_{Kerr}| = 8\pi M_{Kerr}\left(M_{Kerr} + \sqrt{{M_{Kerr}}^2 - L^2_{Kerr}/{M_{Kerr}}^2}\right)$ where M_{Kerr} and L_{Kerr} are respectively the total mass and the total angular momentum of the Kerr spacetime (the angular momentum can be defined also as a suitable integral at infinity). This means that M_{Kerr} is the asymptotic value of the Bondi mass along the future null infinite $\mathscr{I}^+$. Assuming that the Bondi

mass tends to the M_{ADM} of the initial slice, inequality (2.3.5) follows because the Bondi mass cannot increase along the evolution. Moreover, equality holds if and only if Σ is a slice of the Kruskal extension of the Schwarzschild spacetime.

It is important to remark than inequality (2.3.5) is global in the sense that, in order to locate the cut $\mathscr{H}$, it is necessary to know the global structure of the spacetime. Penrose proposed to estimate the area $|\mathscr{H}|$ from below in terms of the area of certain surfaces which can be defined independently of the future evolution of the spacetime. The validity of these estimates relies on the validity of the cosmic censorship. These types of inequalities are collectively called *Penrose inequalities* and they are interesting for several reasons. First of all, they would provide a strengthening of the PMT. Moreover, they would also give indirect support to the validity of cosmic censorship, which is a basic ingredient in their derivation.

There are several versions of the Penrose inequality. Typically one considers closed surfaces S embedded in a spacelike hypersurface with a selected asymptotically flat end Σ_0^∞ which are *bounding* with respect to a suitable large sphere in Σ_0^∞. This leads to the following definition:

Definition 2.3.6 *Consider a spacelike hypersurface* (Σ, g, K) *possibly with boundary with a selected asymptotically flat end* Σ_0^∞*. Take a sphere* $S_b \subset \Sigma_0^\infty$ *with* $r = r_0 = const$ *large enough so that the spheres with* $r \geq r_0$ *are outer untrapped with respect to the direction pointing into the asymptotic region in* Σ_0^∞*. Let* $\Omega_b = \Sigma \setminus \{r > r_0\}$*, which is obviously topologically closed and satisfies* $S_b \subset \partial\Omega_b$*. Then* S_b *is a barrier with interior* Ω_b*. A surface* $S \subset \Sigma$ *will be called* **bounding** *if it is bounding with respect to* S_b*.*

Remark 1. It is well-known that on an asymptotically flat end Σ_0^∞, the surfaces at constant r are, for large enough r, outer untrapped. Essentially, this definition establishes a specific form of selecting the barrier in hypersurfaces containing a selected asymptotically flat end. □

Remark 2. Obviously, the definitions of exterior and interior of a bounding surface (Definition 2.2.26), enclosing (Definition 2.2.27), outermost (Definition 2.2.28) and $T^\pm$ (Definitions 2.2.29 and 2.2.30), given in the previous section, are applicable in the asymptotically flat setting. Moreover, since r_0 can be taken as large as desired, the specific choice of S_b and Ω_b is not relevant for the definition of bounding (once the asymptotically flat end has been selected). Because of that, when considering asymptotically flat ends, we will refer to the exterior of

S in Ω_b as the *exterior of S in* Σ. □

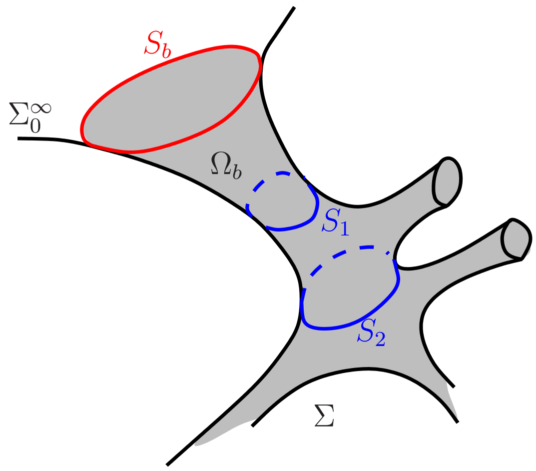


Figure 2.6: The hypersurface Σ possesses an asymptotically flat end Σ_0^∞ but also other types of ends and boundaries. The surface S_b, which represents a large sphere in Σ_0^∞ and is outer untrapped, is a barrier with interior Ω_b (in grey). The surface S_1 is bounding with respect to S_b (c.f. Definition 2.2.26) and therefore is bounding. The surface S_2 fails to be bounding (c.f. Figure 2.4).

The standard version of the Penrose inequality reads

$$M_{ADM} \geq \sqrt{\frac{A_{min}(\partial^{top} T^+)}{16\pi}}, \tag{2.3.6}$$

where $A_{min}(\partial^{top} T^+)$ is the minimal area necessary to enclose $\partial^{top} T^+$. This inequality (2.3.6) is a consequence of the heuristic argument outlined before because (under cosmic censorship) $\mathscr{H}$ encloses $\partial^{top} T^+$ The minimal area enclosure of $\partial^{top} T^+$ needs to be taken because $\mathscr{H}$ could still have less area than $\partial^{top} T^+$ [69].

By reversing the time orientation, the same argument yields (2.3.6) with $\partial^{top} T^+$ replaced by $\partial^{top} T^-$. In general, neither $\partial^{top} T^+$ encloses $\partial^{top} T^-$ nor vice versa. In the case that $K_{ij} = 0$, these inequalities simplify because $T^+ = T^-$ and $\partial^{top} T^+$ is the outermost minimal surface (i.e. a minimal surface enclosing any other bounding minimal surface in Σ) and, hence, its own minimal area enclosure. The inequality in this case is called *Riemannian Penrose inequality* and it has been proven for connected $\partial^{top} T^+$ in [70] and in the general case in [19] using a different method. In the non-time-symmetric case, (2.3.6) is not invariant

under time reversals. Moreover, the minimal area enclosure of a given surface S can be a rather complicated object typically consisting of portions of S together with portions of minimal surfaces outside of S. This complicates the problem substantially. This has led several authors to propose simpler looking versions of the inequality, even if they are not directly supported by cosmic censorship. Two of such extensions are

$$M_{ADM} \geq \sqrt{\frac{A_{min}(\partial^{top}(T^+ \cup T^-))}{16\pi}}, \quad M_{ADM} \geq \sqrt{\frac{|\partial^{top}(T^+ \cup T^-)|}{16\pi}}, \tag{2.3.7}$$

(see e.g. [75]). These inequalities are immediately stronger than (2.3.6) and have the advantage of being invariant under time reversals. The second inequality avoids even the use of minimal area enclosures. Neither version is supported by cosmic censorship and at present there is little evidence for their validity. However, both reduce to the standard version in the Riemannian case and both hold in spherical symmetry. No counterexamples are known either. It would be interesting to have either stronger support for them, or else to find a counterexample.

Recently, Bray and Khuri proposed [20] a new method to approach the general (i.e. non time-symmetric) Penrose inequality. The basic idea was to modify the Jang equation [74], [104] so that the product manifold $\Sigma \times \mathbb{R}$ used to construct the graphs which define the Jang equation is endowed with a warped type metric of the form $-\varphi^2 dt^2 + g$ instead of the product metric. Their aim was to reduce the general Penrose inequality to the Riemannian Penrose inequality on the graph manifold. A discussion on the type of divergences that could possibly occur for the generalized Jang equation led the authors to consider a new type of trapped surfaces which they called **generalized trapped surfaces** and **generalized apparent horizons** (defined in Section 2.2.1). This type of surfaces have very interesting properties. The most notable one is given by Theorem 2.2.32 [53] which guarantees the existence, uniqueness and $C^{2,\alpha}$-regularity of the outermost generalized apparent horizon S_{out}. The Penrose inequality proposed by these authors reads

$$M_{ADM} \geq \sqrt{\frac{|S_{out}|}{16\pi}}, \tag{2.3.8}$$

with equality only if the spacetime is Schwarzschild. This inequality has several remarkable properties that makes it very appealing [20]. First of all, the definition of generalized apparent horizon, and hence the corresponding Penrose inequality, is insensitive to time reversals. Moreover, there is no need of taking the minimal area enclosure of S_{out}, as this surface has less area than any of its enclosures (c.f.

Theorem 2.2.32). Since MOTS are automatically generalized trapped surfaces, S_{out} encloses the outermost MOTS $\partial^{top}T^{+}$. Thus, (2.3.8) is stronger than (2.3.6) and its proof would also establish the standard version of the Penrose inequality. Moreover, Khuri has proven [76] that no generalized trapped surfaces exist in Minkowski, which is a necessary condition for the validity of (2.3.8). Another interesting property of this version, and one of its motivations discussed in [20], is that the equality case in (2.3.8) covers a larger number of slices of Kruskal than the equality case in (2.3.6). Recall that the rigidity statement of any version of the Penrose inequality asserts that equality implies that (Σ, g, K) is a hypersurface of Kruskal. However, *which* slices of Kruskal satisfy the equality case may depend on the version under consideration. The more slices having this property, the more accurate the version can be considered. For any slice Σ of Kruskal we can define Σ^{+} as the intersection of Σ with the domain of outer communications. Bray and Khuri noticed that whenever $\partial^{top}\Sigma^{+}$ intersects both the black hole and the white hole event horizons, then the standard version (2.3.6) gives, in fact, a strict inequality. Although (2.3.8) does not give equality for all slices of Kruskal, it does so in all cases where the boundary of Σ^{+} is a $C^{2,\alpha}$ surface (provided this boundary is the outermost generalized apparent horizon). It follows that version (2.3.8) contains more cases of equality than (2.3.6) and is therefore more accurate. It should be stressed that the second inequality in (2.3.7) gives equality for *all* slices of Kruskal, so in this sense it would be optimal.

Despite its appealing properties, (2.3.8) is *not* directly supported by cosmic censorship. The reason is that the outermost generalized apparent horizon need not always lie inside the event horizon. A simple example [80] is given by a slice Σ of Kruskal such that $\partial^{top}T^{+}$ (which corresponds to the intersection of Σ with the black hole event horizon) and $\partial^{top}T^{-}$ (the intersection Σ with the white hole horizon) meet transversally. Since both surfaces are generalized trapped surfaces, Theorem 2.2.32 implies that there must exist a unique $C^{2,\alpha}$ outermost generalized apparent horizon enclosing both. This surface must therefore penetrate into the exterior region Σ^{+} somewhere, as claimed. We will return to the issue of the Penrose inequality in Chapter 6, where we will find a counterexample of (2.3.8) precisely by studying the outermost generalized apparent horizon in this type of slices in the Kruskal spacetime. For further information about the present status of the Penrose inequality, see [80].

2.4 Uniqueness of Black Holes

According to cosmic censorship, any gravitational collapse that settles down to a stationary state should approach a stationary black hole. The *black hole uniqueness theorems* aim to classify all the stationary black hole solutions of Einstein equations. In this section we will first summarize briefly the status of stationary black hole uniqueness theorems. We will also describe in some detail a powerful method (the so-called *doubling method* of Bunting and Masood-ul-Alam) to prove uniqueness for *static* black holes which will be essential in Chapter 5.

In the late sixties and early seventies the properties of equilibrium states of black holes were extensively studied by many theoretical physicists interested in the gravitational collapse process. The first uniqueness theorem for black holes was found by W. Israel in 1967 [71], who found the very surprising result that a static, topologically spherical vacuum black hole is described by the Schwarzschild solution. In the following years, several works ([89], [100], [23]) established that the Schwarzschild solution indeed exhausts the class of static vacuum black holes with non-degenerate horizons. The method of the proofs in [71], [89], [100] consisted in constructing two integral identities which were used to investigate the geometric properties of the level surfaces of the norm of the static Killing. This method proved uniqueness under the assumption of connectedness and non-degeneracy of the event horizon. The hypothesis on the connectedness of the horizon was dropped by Bunting and Masood-ul-Alam [23] who devised a new method based on finding a suitable conformal rescalling which allowed using the rigidity part of the PMT to conclude uniqueness. This method, known as the *doubling method* is, still nowadays, the most powerful method to prove uniqueness of black holes in the static case. Finally, the hypothesis on the non-degeneracy of the event horizon was dropped by Chruściel [39] in 1999 who applied the doubling method across the non-degenerate components and applied the PMT for complete manifolds with one asymptotically flat end (Theorem 2.4.12 below) to conclude uniqueness (the Bunting and Masood-ul-Alam conformal rescalling transforms the degenerate components into cylindrical ends). The developments in the uniqueness of static electro-vacuum black holes go in parallel to the developments in the vacuum case. Some remarkable works which played an important role in the general proof of the uniqueness of static electro-vacuum black holes are [72], [90], [108], [102], [109], [84], [40], [45]. Uniqueness of static black holes using the doubling method has also been proved for other matter models, as for instance the Einstein-Maxwell-dilaton model [85], [83].

During the late sixties, uniqueness of *stationary* black holes also started to take shape. In fact, the works of Israel, Hawking, Carter and Robinson, between 1967 and 1975, gave an almost complete proof that the Kerr black hole was the only possible stationary vacuum black hole. The first step was given by Hawking (see [65]) who proved that the intersection of the event horizon with a Cauchy hypersurface has $\mathbb{S}^2$-topology. The next step, also due to Hawking [65] was the demonstration of the so-called Hawking Rigidity Theorem, which states that a stationary black hole must be static or axisymmetric. Finally, the work of Carter [33] and Robinson [99] succeeded in proving that the Kerr solutions are the only possible stationary axisymmetric black holes. Nevertheless, due to the fact that the Hawking Rigidity Theorem requires analyticity of all objects involved, uniqueness was proven only for analytic spacetimes. The recent work [42] by Chruściel and Lopes Costa has contributed substantially to reduce the hypotheses and to fill several gaps present in the previous arguments. Similarly, uniqueness of stationary electro-vacuum black holes has been proven for analytic spacetimes. Some remarkable works for the stationary electro-vacuum case are [34], [86] and, more recently, [49], where weaker hypotheses are assumed for the proof. Uniqueness of stationary and axisymmetric black holes has also been proven for non-linear σ-models in [22]. The Hawking Rigidity Theorem has not been generalized to non-linear σ-models and, hence, axisymmetry is required in this case. It is also worth to remark that, in the case of matter models modeled with Yang-Mills fields, uniqueness of stationary black holes is not true in general and counterexamples exist [11].

In this thesis we will be interested in uniqueness theorems for static *quasi-local* black holes and, particularly, in the doubling method of Bunting and Masood-ul-Alam. In the remainder of this chapter, we will describe this method in some detail by giving a sketch of the proof of the uniqueness theorem for static electro-vacuum black holes.

2.4.1 Example: Uniqueness for electro-vacuum static black holes

Let us start with some definitions. An electro-vacuum solution of the Einstein field equations is a triad $(M, g^{(4)}, \mathbf{F})$, where $\mathbf{F}$ is the source-free electromagnetic tensor, i.e. a 2-form satisfying the Maxwell equations which no sources, i.e.

$$\nabla^{\mu} F_{\mu\nu} = 0,$$
$$\nabla_{[\alpha} F_{\mu\nu]} = 0,$$

and $(M, g^{(4)})$ is the spacetime satisfying the Einstein equations with energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\alpha}F_{\nu}{}^{\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}g^{(4)}_{\mu\nu}\right).$$

We call a stationary electro-vacuum spacetime an electro-vacuum spacetime admitting a stationary Killing vector field $\vec{\xi}$, satisfying $\mathcal{L}_{\vec{\xi}}F_{\mu\nu} = 0$. Let us define the electric and magnetic fields with respect to $\vec{\xi}$ as

$$\begin{aligned} E_\mu &= -F_{\mu\nu}\xi^\nu, \\ B_\mu &= (*F)_{\mu\nu}\xi^\nu, \end{aligned}$$

respectively. Here, $*\mathbf{F}$ denotes the Hodge dual of $\mathbf{F}$ defined as

$$(*F)_{\mu\nu} = \frac{1}{2}\eta^{(4)}_{\mu\nu\alpha\beta}F^{\alpha\beta}.$$

From the Maxwell equations and $\mathcal{L}_{\vec{\xi}}F_{\mu\nu} = 0$ it follows easily that $d\mathbf{E} = 0$ and $d\mathbf{B} = 0$ which implies that, at least locally, there exist two functions ϕ and ψ, called the **electric** and **magnetic potentials**, so that $\mathbf{E} = -d\phi$ and $\mathbf{B} = -d\psi$, respectively. These potentials are defined up to an additive constant and they satisfy $\vec{\xi}(\phi) = \vec{\xi}(\psi) = 0$.

Definition 2.4.1 *A stationary electro-vacuum spacetime* $(M, g^{(4)}, \mathbf{F})$ *with Killing field* $\vec{\xi}$ *is said to be* **purely electric** *with respect to* $\vec{\xi}$ *if and only if* $\mathbf{B} = 0$.

For simplicity, we will restrict ourselves to the purely electric case. In fact, the general case can be reduced to the purely electric case by a transformation called *duality rotation* [66].

In the static case there exists an important simplification which allows to reduce the formulation of the uniqueness theorem for black holes in terms of conditions on a spacelike hypersurface instead of conditions on the spacetime. The fact is that, under suitable circumstances, the presence of an event horizon in a static spacetime implies the existence of an asymptotically flat hypersurface with compact topological boundary such that the static Killing field is causal everywhere and null precisely on the boundary. Then, the uniqueness theorem for static electro-vacuum black holes can be stated simply as follows.

Theorem 2.4.2 (Chruściel, Tod, 2006 [45]) *Let* $(M, g^{(4)}, F)$ *be a static solution of the Einstein-Maxwell equations. Suppose that* M *contains a simply connected asymptotically flat hypersurface* Σ *with non-empty topological boundary such that* Σ *is the union of an asymptotically flat end and a compact set, such that:*

- *The topological boundary $\partial^{top}\Sigma$ is a compact, 2-dimensional embedded topological submanifold.*

- *The static Killing vector field is causal on Σ and null only on $\partial^{top}\Sigma$.*

Then, after performing a duality rotation of the electromagnetic field if necessary:

- *If $\partial^{top}\Sigma$ is connected, then Σ is diffeomorphic to $\mathbb{R}^3$ minus a ball. Moreover, there exists a neighbourhood of Σ in M which is isometrically diffeomorphic to an open subset of the Reissner-Nordström spacetime.*

- *If $\partial^{top}\Sigma$ is not connected, then Σ is diffeomorphic to $\mathbb{R}^3$ minus a finite union of disjoint balls and there exists a neighborhood of Σ in M which is isometrically diffeomorphic to an open subset of the standard Majumdar-Papapetrou spacetime.*

Remark. The standard Majumdar-Papapetrou spacetime is the manifold $(\mathbb{R}^3 \setminus \bigcup_{i=1}^{n} \mathfrak{p}_i) \times \mathbb{R}$ endowed with the metric $ds^2 = \frac{-dt^2}{u^2} + u^2(dx^2 + dy^2 + dz^2)$, where $u = 1 + \sum_{i=1}^{n} \frac{q_i}{r_i}$ with q_i being a constant and r_i the Euclidean distance to $\mathfrak{p}_i$. $\Box$

In what follows we will give a sketch of the proof of the Theorem 2.4.2. Firstly, we need some results concerning the boundary of the set $\{\mathfrak{p} \in M : \lambda|_{\mathfrak{p}} > 0\}$, where $\lambda \equiv -\xi_\mu \xi^\mu$, i.e. minus the squared norm of the stationary Killing field $\vec{\xi}$.

Let us start with some definitions.

Definition 2.4.3 *Let $(M, g^{(4)})$ be a spacetime with a Killing vector $\vec{\xi}$. A* **Killing prehorizon** *$\mathcal{H}_{\vec{\xi}}$ of $\vec{\xi}$ is a null, 3-dimensional submanifold (not necessarily embedded), at least C^1, such that $\vec{\xi}$ is tangent to $\mathcal{H}_{\vec{\xi}}$, null and different from zero.*

Definition 2.4.4 *A* **Killing horizon** *is an embedded Killing prehorizon.*

Next, let us introduce a quantity κ defined on a Killing prehorizon in any stationary spacetime. Clearly, on a Killing prehorizon $\mathcal{H}_{\vec{\xi}}$ we have $\lambda = 0$. It implies that $\nabla_\mu \lambda$ is normal to $\mathcal{H}_{\vec{\xi}}$. Now, since $\vec{\xi}$ is null and tangent to $\mathcal{H}_{\vec{\xi}}$, it is also normal to $\mathcal{H}_{\vec{\xi}}$. Since, moreover $\vec{\xi}\big|_{\mathcal{H}_{\vec{\xi}}}$ is nowhere zero, it follows that there exists a function κ such that

$$\nabla_\mu \lambda = 2\kappa \xi_\mu. \tag{2.4.1}$$

κ is called the **surface gravity** on $\mathcal{H}_{\vec{\xi}}$. The following result states the constancy of κ on a Killing prehorizon in a static spacetime.

Lemma 2.4.5 (Rácz, Wald, 1996 [98]) *Let $\mathcal{H}_{\vec{\xi}}$ be a Killing prehorizon for an integrable Killing vector $\vec{\xi}$. Then κ is constant on each arc-connected component of $\mathcal{H}_{\vec{\xi}}$.*

Remark. This lemma also holds in stationary spacetimes provided the DEC holds. Its proof can be found in Chapter 12 of [112]. □

This lemma allows to classify Killing prehorizons in static spacetimes in two types with very different behavior.

Definition 2.4.6 *An arc-connected Killing prehorizon $\mathcal{H}_{\vec{\xi}}$ is called* **degenerate** *when $\kappa = 0$ and* **non-degenerate** *when $\kappa \neq 0$.*

Since $\nabla_\mu \lambda \neq 0$ on a non-degenerate Killing prehorizon, the set $\{\lambda = 0\}$ defines an embedded submanifold (c.f. [41]).

Lemma 2.4.7 *Non-degenerate Killing prehorizons are Killing horizons.*

The next lemma guarantees the existence of a Killing prehorizon in a static spacetime. This lemma will be used several times along this thesis. For completeness, we find it appropriate to include its proof (we essentially follow [39]).

Lemma 2.4.8 (Vishveshwara, 1968 [111], Carter, 1969 [32]) *Let $(M, g^{(4)})$ be a static spacetime with Killing vector $\vec{\xi}$. Then the set $\mathcal{N}_{\vec{\xi}} \equiv \partial^{top}\{\lambda > 0\} \cap \{\vec{\xi} \neq 0\}$, if non-empty, is a smooth Killing prehorizon.*

Proof. Consider a point $\mathfrak{p} \in \mathcal{N}_{\vec{\xi}}$. Due to the Fröbenius's theorem (see e.g. [78]), staticity implies that there exists a neighbourhood $\mathcal{V}_0 \subset M$ of $\mathfrak{p}$, with $\vec{\xi}\big|_{\mathcal{V}_0} \neq 0$, which (for $\mathcal{V}_0$ small enough) is foliated by a family of smooth embedded submanifolds Σ_t of codimension one and orthogonal to $\vec{\xi}$. In particular, $\mathfrak{p} \in \Sigma_0$, where Σ_0 denotes a leaf of this foliation.

Now consider the leaves Σ_α of the Σ_t foliation such that $\Sigma_\alpha \cap \{\lambda \neq 0\} \neq \emptyset$. The staticity condition (2.1.3) implies

$$\xi_{[\nu} \nabla_{\mu]} \lambda = \lambda \nabla_{[\mu} \xi_{\nu]},$$

which on $\mathcal{V}_0 \cap \{\lambda \neq 0\}$ reads

$$\xi_{[\nu} \nabla_{\mu]} (\ln |\lambda|) = \nabla_{[\mu} \xi_{\nu]}. \tag{2.4.2}$$

Let $\vec{W}$ and $\vec{Z}$ be smooth vector fields on $\mathcal{V}_0$ such that $\vec{W}$ satisfies $\xi_\mu W^\mu = 1$ and $\vec{Z}$ is tangent to the leaves Σ_t. At points of Σ_α on which $\lambda \neq 0$, the contraction of equation (2.4.2) with $Z^\mu W^\nu$ gives

$$Z^\mu \nabla_\mu (\ln |\lambda|) = 2 Z^\mu W^\nu \nabla_{[\mu} \xi_{\nu]}.$$

The right-hand side of this equation is uniformly bounded on Σ_α, which implies that $\ln |\lambda|$ is uniformly bounded on $\Sigma_\alpha \cap \{\lambda \neq 0\}$. This is only possible if $\Sigma_\alpha \cap \{\lambda = 0\} = \emptyset$. Consequently, λ is either positive, or negative, or zero in each leaf of the foliation Σ_t. In particular, it implies that $\{\lambda = 0\} \cap \mathcal{V}_0$ is a union of leaves of the Σ_t foliation.

It only remains to prove that each arc-connected component of $\partial^{top}\{\lambda > 0\} \cap \mathcal{V}_0$ coincides with one of these leaves. For that, take coordinates $\{z, x^A\}$ in $\mathcal{V}_0$ in such a way that the coordinate z characterizes the leaves of the foliation Σ_t and $\mathfrak{p} = (z = 0, x^A = 0)$ (this is possible because each leaf of Σ_t is an embedded submanifold of $\mathcal{V}_0$). Note that the leaf $\Sigma_0 \ni \mathfrak{p}$ is then defined by $\{z = 0\}$. In this setting, we just need to prove that $\{z = 0\}$ coincides with an arc-connected component of $\partial^{top}\{\lambda > 0\} \cap \mathcal{V}_0$. Due to the fact that $\mathfrak{p} \in \partial^{top}\{\lambda > 0\} \cap \mathcal{V}_0$, there exists a sequence of points $\mathfrak{p}_i \in \mathcal{V}_0$ with $\lambda > 0$ which converge to $\mathfrak{p}$ and have coordinates $(z(\mathfrak{p}_i), x^A(\mathfrak{p}_i))$. Since the coordinate z characterizes the leaves and λ is either positive, or negative, or zero in each leaf, it follows that the sequence of points $\mathfrak{p}'_i$ with coordinates $(z(\mathfrak{p}_i), 0)$ also has $\lambda > 0$ and tends to $\mathfrak{p}$. By the same reason, given any point $\mathfrak{q} \in \{z = 0\}$ with coordinates $(0, x_0^A)$, the sequence of points $\mathfrak{q}_i = (z(\mathfrak{p}_i), x_0^A)$ tends to $\mathfrak{q}$ and lies in $\{\lambda > 0\}$. Therefore, $\{z = 0\}$ is composed precisely by the points of the arc-connected component of $\partial^{top}\{\lambda > 0\} \cap \mathcal{V}_0$ which contains $\mathfrak{p}$. This implies that every arc-connected component of $\partial^{top}\{\lambda > 0\} \cap \mathcal{V}_0$ coincides with a leaf Σ_t where $\lambda \equiv 0$ (and $\vec{\xi} \neq 0$). Finally, this local argument can be extended to the whole set $\mathcal{N}_{\vec{\xi}}$ simply by taking a covering of $\mathcal{N}_{\vec{\xi}}$ by suitable open neighbourhoods $\mathcal{V}_\beta \subset M$. ■

Remark. Although each arc-connected component of $\partial^{top}\{\lambda > 0\} \cap \mathcal{V}_\beta$ is an embedded submanifold of $\mathcal{V}_\beta \subset M$, the whole set $\mathcal{N}_{\vec{\xi}}$ may fail to be embedded in M (see Figure 2.7). Thus, a priori, degenerate Killing prehorizons may fail to be embedded. As mentioned before, this possibility has been overlooked in the literature until recently [41]. The occurrence of non-embedded Killing prehorizons poses serious difficulties for the uniqueness proofs. One way to deal with these objects is to make hypotheses that simply exclude them. In Proposition 2.4.11 below, the hypothesis that $\partial^{top}\Sigma$ is a compact and embedded

topological manifold is made precisely for this purpose. Another possibility is to prove that these prehorizons do not exist. At present, this is only known under strong global hypotheses on the spacetime (c.f. Definition 2.4.14 below). It is an interesting open problem to either find an example of a non-embedded Killing prehorizon or else to prove that they do not exist. □

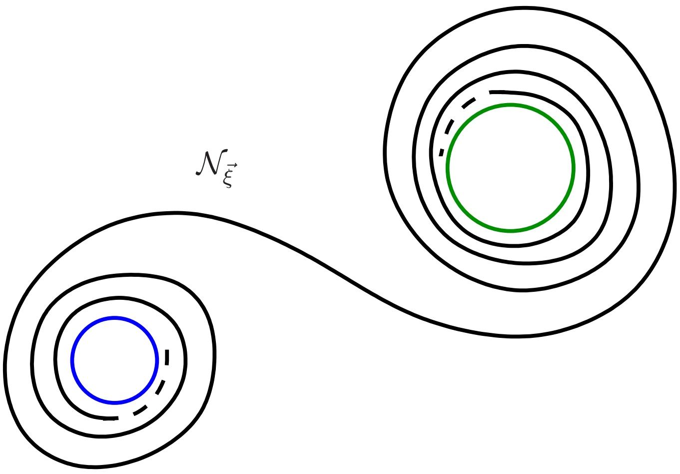


Figure 2.7: The figure illustrates a situation where $\mathcal{N}_{\vec{\xi}} = \partial^{top}\{\lambda > 0\} \cap \{\vec{\xi} \neq 0\}$ fails to be embedded. In this figure, the Killing vector is nowhere zero, causal everywhere and null precisely on the plotted line. Here, $\mathcal{N}_{\vec{\xi}}$ has three arc-connected components: two spherical and one with spiral form. The fact that the spiral component accumulates around the spheres implies that the whole set $\mathcal{N}_{\vec{\xi}}$ is not embedded. Moreover, the spiral arc-connected component, which is itself embedded, is not compact.

The hypotheses of Theorem 2.4.2 require the existence of a hypersurface Σ with topological boundary such that $\lambda \geq 0$ everywhere and $\lambda = 0$ precisely on $\partial^{top}\Sigma$. It is clear then that $\partial^{top}\Sigma \subset \partial^{top}U$, where $U \equiv \{\mathfrak{p} \in M : \lambda|_{\mathfrak{p}} > 0\}$, but, in general, $\partial^{top}\Sigma$ will not lie in a Killing prehorizon because it can still happen that $\vec{\xi} = 0$ on a subset of $\partial^{top}U$. However, the set of points where $\vec{\xi} = 0$ cannot be very "large" as the next result guarantees.

Theorem 2.4.9 (Boyer, 1969 [17], Chruściel, 1999 [39]) *Consider a static spacetime $(M, g^{(4)})$ with Killing vector $\vec{\xi}$. Let $\mathfrak{p} \in \partial^{top}\{\lambda > 0\}$ be a fixed point (i.e. $\vec{\xi}\,\big|_{\mathfrak{p}} = 0$). Then $\mathfrak{p}$ belongs to a connected, spacelike, smooth, totally geodesic, 2-dimensional surface S_0 which is composed by fixed points. Furthermore, S_0 lies in the closure of a non-degenerate Killing horizon $\mathcal{H}_{\vec{\xi}}$*

Therefore, using Lemma 2.4.8 and Theorem 2.4.9, we can assert that $\partial^{top}\{\lambda > 0\}$ *belongs to the closure of a Killing prehorizon.*

The manifold $\text{int}(\Sigma)$ admits, besides the induced metric, a second metric h called *orbit space metric* which is a key object in the uniqueness proof. Let us first define the projector orthogonal to $\vec{\xi}$.

Definition 2.4.10 *On the open set $U \equiv \{\lambda > 0\} \subset M$, the* **projector orthogonal** *to $\vec{\xi}$, denoted by $h_{\mu\nu}$, is defined as*

$$h_{\mu\nu} \equiv g^{(4)}_{\mu\nu} + \frac{\xi_\mu \xi_\nu}{\lambda}. \tag{2.4.3}$$

This tensor has the following properties:

- *It is symmetric, i.e. $h_{\mu\nu} = h_{\nu\mu}$.*
- *It has rank 3.*
- *It satisfies $h_{\mu\nu}\xi^\mu = 0$*

On U we can also define the function $V = +\sqrt{\lambda}$. The hypersurface $\text{int}(\Sigma)$ is fully contained in U. Let $\Phi : \text{int}(\Sigma) \to U \subset M$ denote the embedding of $\text{int}(\Sigma)$ in U, then the pull-back of the projector $\Phi^*(h)$ is a Riemannian metric on Σ. We will denote by the same symbols h, V and ϕ both the objects in $U \subset M$ and their corresponding pull-backs in $\text{int}(\Sigma)$.

The Einstein-Maxwell field equations for a purely electric stationary electro-vacuum spacetime are equivalent to the following equations on $\text{int}(\Sigma)$ see e.g. [67].

$$V\Delta_h \phi = D_i V D^i \phi, \tag{2.4.4}$$
$$V\Delta_h V = D_i \phi D^i \phi, \tag{2.4.5}$$
$$V R_{ij}(h) = D_i D_j V + \frac{1}{V}\left(D_k \phi D^k \phi h_{ij} - 2 D_i \phi D_j \phi\right), \tag{2.4.6}$$

where D and $R_{ij}(h)$ are the covariant derivative and the Ricci tensor of the Riemannian metric h, respectively. Indices are raised and lowered with h_{ij} and its inverse h^{ij}.

In the asymptotically flat end Σ_0^∞ of $\text{int}(\Sigma)$, the Einstein equations on $\text{int}(\Sigma)$ and (2.3.1) that V and ϕ decay as

$$V = 1 - \frac{M_{ADM}}{r} + O^{(2)}(1/r^2), \qquad \phi = \frac{Q}{r} + O^{(2)}(1/r^2), \tag{2.4.7}$$

where Q is a constant (called the **electric charge** associated with Σ_0^∞), and M_{ADM} is the corresponding ADM mass.

A crucial step for the uniqueness proof is to understand the behavior of the Riemannian metric h near the boundary $\partial^{top}\Sigma$. This is the aim of the following proposition.

Proposition 2.4.11 (Chruściel, 1999 [39]) *Let Σ be a spacelike hypersurface in a static spacetime $(M, g^{(4)})$ with Killing vector $\vec{\xi}$. Suppose that $\lambda \geq 0$ on Σ with $\lambda = 0$ precisely on its topological boundary $\partial^{top}\Sigma$ which is assumed to be a compact, 2-dimensional and embedded topological manifold. Then*

1. *Every arc-connected component $(\partial^{top}\Sigma)_d$ which intersects a C^2 degenerate Killing horizon corresponds to a complete cylindrical asymptotic end of (Σ, h).*

2. *$(\overline{\Sigma}, h)$ admits a differentiable structure such that every arc-connected component $(\partial^{top}\Sigma)_n$ of $\partial^{top}\Sigma$ which intersects a non-degenerate Killing horizon is a totally geodesic boundary of (Σ, h) with h being smooth up to and including the boundary.*

This proposition shows that the Riemannian manifold $(\overline{\Sigma} \setminus \bigcup_d (\partial^{top}\Sigma)_d, h)$ is the union of asymptotically flat ends, complete cylindrical asymptotic ends and compact sets with totally geodesic boundaries. Let us define $\tilde{\Sigma} \equiv \overline{\Sigma} \setminus \bigcup_d (\partial^{top}\Sigma)_d$.

Now we are ready to explain the doubling method itself. Recall that the final aim is to show that the spacetime is either Reissner-Nordström or Majumdar-Papapetrou. Both have the property that $(\tilde{\Sigma}, h)$ is conformally flat (i.e. there exists a positive function Ω, called the conformal factor, such that the metric $\Omega^2 h$ is the flat metric). Moreover, conformal flatness together with sufficient information on the conformal factor would imply, via the Einstein field equations, that the spacetime is in fact Reissner-Nordström or Majumdar-Papapetrou.

A powerful method to prove that a given metric is flat is by using the rigidity part of the PMT. Unfortunately Theorem 2.3.5 cannot be applied directly to $(\tilde{\Sigma}, h)$ because, first, $\tilde{\Sigma}$ is a manifold with boundary, and second, $(\tilde{\Sigma}, h)$ has in general cylindrical asymptotic ends and therefore it is not asymptotically flat.

The presence of boundaries was dealt with by Bunting and Masood-ul-Alam who invented a method which constructs a new manifold without boundary to which the PMT can be applied.

To simplify the presentation, let us assume for a moment that $(\tilde{\Sigma}, h)$ has no cylindrical ends, so this manifold is the union of asymptotically ends and a

compact interior with totally geodesic boundaries (by Proposition 2.4.11). Next, find two conformal factors $\Omega_+ > 0$ and $\Omega_- > 0$ such that

- $h_+ \equiv \Omega_+^2 h$ is asymptotically flat, has vanishing mass and $R(h_+) \geq 0$, where $R(h_+)$ is the scalar curvature of h_+.
- $h_- \equiv \Omega_-^2 h$ admits a one point (let us denote it by Υ) compactification of the asymptotically flat infinity, and $R(h_-) \geq 0$.

Then the idea is to glue the manifolds $(\tilde{\Sigma}, h_+)$ and $(\tilde{\Sigma} \cup \Upsilon, h_-)$ across the boundaries to produce a complete, asymptotically flat manifold $(\hat{\Sigma}, \hat{h})$ with no boundaries, vanishing mass and non-negative scalar curvature $\hat{R} \geq 0$. In order to glue the two manifolds with sufficient differentiability, the following two conditions are required:

- $\Omega_+|_{\partial\tilde{\Sigma}} = \Omega_-|_{\partial\tilde{\Sigma}}$,
- $\vec{m}(\Omega_+)|_{\partial\tilde{\Sigma}} = -\,\vec{m}(\Omega_-)|_{\partial\tilde{\Sigma}}$.

where $\vec{m}$ is the unit normal pointing to the interior $\tilde{\Sigma}$ in each of the copies.

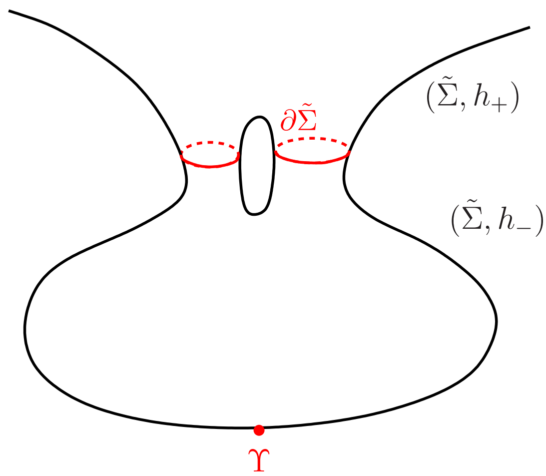


Figure 2.8: The doubled manifold $(\hat{\Sigma}, \hat{h})$ resulting from gluing $(\tilde{\Sigma}, h_+)$ and $(\tilde{\Sigma} \cup \Upsilon, h_-)$.

Theorem 2.3.5 can be applied to $(\hat{\Sigma}, \hat{h})$ to conclude that this space is in fact Euclidean.

When the spacetime also has degenerate horizons the doubling method across non-degenerate components can still be done. The resulting manifold however is no longer asymptotically flat since it contains asymptotically cylindrical ends,

so Theorem 2.3.5 cannot be applied directly. Fortunately, there exists a suitable generalization of the PMT that covers this case. The precise statement is the following.

Theorem 2.4.12 (Bartnik, Chruściel, 1998 [10]) *Let $(\hat{\Sigma}, \hat{h})$ be a smooth complete Riemannian manifold with an asymptotically flat end $\hat{\Sigma}_0^\infty$ and with a smooth one-form $\hat{\mathbf{E}}$ satisfying $\hat{D}_i \hat{E}^i = 0$ and $\hat{E}_i dx^i = \frac{\hat{Q}}{r^2} dr + o(\frac{1}{r^2})$ in $\hat{\Sigma}_0^\infty$, where $\hat{Q}$ is a constant called electric charge. Suppose that $\hat{h}$ satisfies $R(\hat{h}) \geq 2\hat{E}_i \hat{E}^i$ and that*

$$\int_{\hat{\Sigma}_0^\infty} \left(R(\hat{h}) - 2\hat{E}_i \hat{E}^i \right) \eta_{\hat{h}} < \infty.$$

Then the ADM mass $\hat{M}_{ADM}$ of $\hat{\Sigma}_0^\infty$ satisfies $\hat{M}_{ADM} \geq |\hat{Q}|$ and equality holds if and only if locally $\hat{h} = u^2(dx^2 + dy^2 + dz^2)$, $\hat{\mathbf{E}} = \frac{du}{u}$ and $\Delta_\delta u = 0$.

Remark. As a consequence of this result, it is no longer necessary to require that $(\tilde{\Sigma}, h_-)$ admits a one-point compactification. It is only necessary to assume that $(\overline{\Sigma_0^\infty}, h_-)$ is complete. □

It is clear from the discussions above that the key to prove Theorem 2.4.2 is to find suitable conformal factors which allow to conclude that $(\tilde{\Sigma}, h)$ is conformally flat. For the static electro-vacuum case, two conformal factors have been considered, one due to Ruback [102], $\Omega_\pm = \frac{1 \pm V + \phi}{2}$, and another proposed by Masood-ul-Alam [84], $\Omega_\pm = \frac{(1\pm V)^2 - \phi^2}{4}$. Recently, Chruściel has showed [40] that the Ruback conformal factor is the only one which works when degenerate Killing horizons are allowed a priori.

We will therefore consider only the Ruback conformal factors $\Omega_\pm = \frac{1 \pm V + \phi}{2}$. The first thing to do is to check that $\Omega_\pm$ are strictly positive on $\tilde{\Sigma}$. This was shown by Ruback [102] and extended by Chruściel [40] and Chruściel and Tod [45] when there are degenerate horizons.

Proposition 2.4.13 (Ruback, 1988, Chruściel, 1998, Chruściel, Tod, 2006) *On $\tilde{\Sigma}$ it holds $|\phi| \leq 1 - V$. Moreover, equality at one point only occurs when the spacetime is the standard Majumdar-Papapetrou spacetime.*

This proposition implies $\Omega_- > 0$ unless we have Majumdar-Papapetrou. Moreover, since $V \geq 0$ on $\tilde{\Sigma}$, we have $\Omega_+ \geq \Omega_- > 0$ except for the standard Majumdar-Papapetrou.

The remaining ingredients are as follows:

- The matching conditions for the gluing procedure follow easily from the fact that $V|_{\partial\tilde{\Sigma}} = 0$, which immediately implies $\Omega_+|_{\partial\tilde{\Sigma}} = \Omega_-|_{\partial\tilde{\Sigma}}$ and $\vec{m}(\Omega_+)|_{\partial\tilde{\Sigma}} = -\vec{m}(\Omega_-)|_{\partial\tilde{\Sigma}}$.

- The asymptotically flat end (Σ_0^∞) becomes a complete end with respect to the metric h_-. This follows from the asymptotic form $\Omega_- = \frac{1}{4r}(M_{ADM} - Q) + O(1/r^2)$ and the fact that $M_{ADM} > |Q|$ which follows from the positivity of Ω_-.

- The field $\mathbf{E}_\pm \equiv \frac{-(1+\phi)d\phi + VdV}{V(1+\phi\pm V)}$ has the following asymptotic behavior

$$\mathbf{E}_+ = \frac{1}{2}\frac{M_{ADM}+Q}{r^2}dr + o(1/r^2),$$

  and satisfies, from the Einstein field equations, that $D_i^\pm E_\pm{}^i = 0$ and $R(h_\pm) = 2E_\pm^i E_{\pm i}$, where $R(h_\pm)$ is the scalar curvature of $h_\pm$.

- A direct computation gives that the ADM mass and the electric charge of $(\hat{\Sigma}, \hat{h})$ satisfy,

$$\hat{M}_{ADM} = \hat{Q}.$$

Therefore, the rigidity part of Theorem 2.4.12 can be applied, to conclude $\hat{h} = u^2 g_E$, where u is a specific function of (V, ϕ) and g_E is the Euclidean metric. Consequently, h (which was conformally related with $\hat{h}$) is conformally flat. The original proof used at this point the explicit form of $u(\phi, V)$ together with the field equations to conclude that $(\tilde{\Sigma}, h)$ corresponds to the metric of the $\{t = 0\}$ slice of Reissner-Nordström spacetime with $M > |Q|$. This last step has been simplified recently by González and Vera in [61] who show that the Reissner-Nordström and the Majumdar-Papapetrou spacetimes are indeed the only static electro-vacuum spacetimes for which $(\tilde{\Sigma}, h)$ is asymptotically flat and conformally flat.

Summarizing, we have obtained that, in the case when Theorem 2.4.12 can be applied, the spacetime is Reissner-Nordström, and in the cases when it cannot be applied the spacetime is already the standard Majumdar-Papapetrou spacetime. We conclude then that a static and electro-vacuum spacetime corresponding to a black hole must be either the Reissner-Nordström spacetime (where $\partial^{top}\Sigma$ is connected) or the standard Majumdar-Papapetrou spacetime (where $\partial^{top}\Sigma$ is non-connected), which proves Theorem 2.4.2.

Remark. The compactness assumption for the embedded topological submanifold $\partial^{top}\Sigma$ is used in order to ensure that $(\hat{\Sigma}, \hat{h})$ is complete. It would be

interesting to study whether this condition can be relaxed or not. □

We will finish this chapter by giving a brief discussion about the global approach of Theorem 2.4.2. In several works ([39], [41] and [44]) Chrúsciel and Galloway have studied sufficient hypotheses which ensure that a black hole spacetime possesses a spacelike hypersurface Σ like the one required in Theorem 2.4.2 and, also, which assumptions are needed to conclude uniqueness for the whole spacetime (or at least for the domain of outer communications) The first work on the subject, namely [39], deals with the vacuum case and requires, among other things, the spacetime to be analytic (although this hypothesis was not explicitly mentioned in [39] and it was included only in the correction [41]). This hypothesis is needed to avoid the existence of non-embedded degenerate Killing prehorizons, which implies that $\partial^{top}\Sigma$ may fail to be compact and embedded as required in Theorem 2.4.2. In [41], Chruściel was able to drop the analyticity assumption by assuming a second Killing vector on M generating a $U(1)$ action and a global hypothesis (named I^+-regularity in the later paper [42]). Finally, in [44] the assumption on the existence of a second Killing field was removed and the result was explicitly extended to the electro-vacuum case. Before giving the statement of such a result, let us define the property of I^+-regularity of a spacetime.

Definition 2.4.14 *Let $(M, g^{(4)})$ be a stationary spacetime containing an asymptotically flat end and let $\vec{\xi}$ be the stationary Killing vector field on M. $(M, g^{(4)})$ is I^+-**regular** if $\vec{\xi}$ is complete, if the domain of outer communications M_{DOC} is globally hyperbolic, and if M_{DOC} contains a spacelike, connected, acausal hypersurface Σ containing an asymptotically flat end, the closure $\overline{\Sigma}$ of which is a C^0 manifold with boundary, consisting of the union of a compact set and a finite number of asymptotically flat ends, such that $\partial^{top}\Sigma$ is an embedded surface satisfying*

$$\partial^{top}\Sigma \subset \mathcal{E}^+ \equiv \partial^{top} M_{DOC} \cap I^+(M_{DOC}),$$

with $\partial^{top}\Sigma$ intersecting every generator of $\mathcal{E}^+$ just once.

Then the result by Chruściel and Galloway states the following.

Theorem 2.4.15 (Chruściel and Galloway, 2010 [44]) *Let $(M, g^{(4)})$ be a static solution of the electro-vacuum Einstein equations. Assume that $(M, g^{(4)})$ is I^+-regular. Then the conclusions of Theorem 2.4.2 hold. Moreover, M_{DOC} is isometrically diffeomorphic to the domain of outer communications of either the Reissner-Nordström spacetime or the standard Majumdar-Papapetrou spacetime.*

Chapter 3

Stability of marginally outer trapped surfaces and symmetries

3.1 Introduction

As we have already mentioned in Chapter 1, although the main aim of this thesis is to study properties of certain types of trapped surfaces, specially weakly outer trapped surfaces and MOTS, in stationary and static configurations, isometries are not the only type of symmetries which can be involved in physical situations of interest. For instance, many relevant spacetimes admit other types of symmetries, such as conformal symmetries, e.g. in Friedmann-Lemaître-Robertson-Walker (FLRW) cosmological models. Another interesting example appears when studying the critical collapse, which is a universal feature of many matter models. Indeed, the critical solution, which separates those configurations that disperse from those that form black holes, are known to admit either a continuous or a discrete self-similarity. Therefore, it is interesting to understand the relationship between trapped surfaces and several special types of symmetries. This is precisely the aim of this chapter.

A recent interesting example of this interplay has been given in [14], [15], [16] where the location of the boundaries of the spacetime set containing weakly trapped surfaces and weakly outer trapped surfaces was analyzed, firstly, in the Vaidya spacetime [14], [15] (which is one of the simplest dynamical situations) and, later, in spherically symmetric spacetimes in general [16]. In these analyses the presence of symmetries turned out to be fundamental. In the important case of isometries, general results on the relationship between weakly trapped surfaces and Killing vectors were discussed in [82], where the first variation of area was used to obtain several restrictions on the existence of weakly trapped surfaces in spacetime regions possessing a causal Killing vector. More specifically, weakly

trapped surfaces can exist in the region where the Killing vector is timelike only if their mean curvature vanishes identically. By obtaining a general identity for the first variation of area in terms of the deformation tensor of an arbitrary vector (defined in equation (2.1.2)), similar restrictions were obtained for spacetimes admitting other types of symmetries, such as conformal Killing vectors or Kerr-Schild vectors (see [47] for its definition). The same idea was also applied in [107] to obtain analogous results in spacetimes with vanishing curvature invariants. The interplay between isometries and dynamical horizons (which are spacelike hypersurfaces foliated by marginally trapped surfaces) was considered in [6] where it was proven that dynamical horizons cannot exist in spacetime regions containing a nowhere vanishing causal Killing vector, provided the spacetime satisfies the NEC. Regarding MOTS, the relation between stable MOTS and isometries was considered in [3], where it was shown that, given a strictly stable MOTS S in a hypersurface Σ (not necessarily spacelike), any Killing vector on S tangent to Σ must in fact be tangent to S.

In the present chapter, we will study the interplay between stable and outermost properties of MOTS in spacetimes possessing special types of vector fields $\vec{\xi}$, including isometries, homotheties and conformal Killing vectors. In fact, we will find results involving completely general vector fields $\vec{\xi}$ and then, we will particularize them to the different types of symmetries. More precisely, we will find restrictions on $\vec{\xi}$ on stable, strictly stable and locally outermost MOTS S in a given spacelike hypersurface Σ, or alternatively, forbid the existence of a MOTS in certain regions where $\vec{\xi}$ fails to satisfy those restrictions. In what follows, we give a brief summary of the present chapter.

The fundamental idea which will allow us to obtain the results of this chapter will be introduced in Section 3.2. As we will see, it will consist in a geometrical construction which can potentially restrict a vector field $\vec{\xi}$ on the outermost MOTS S. The geometrical procedure will involve the analysis of the stability operator $L_{\vec{m}}$ of a MOTS acting on a certain function Q. It will turn out that the results obtained by the geometric construction can, in most cases, be sharpened considerably by using the maximum principle of elliptic operators. This will also allow us to extend the validity of the results from the outermost case to the case of stable and strictly stable MOTS. However, the defining expression (2.2.11) for the stability operator $L_{\vec{m}}Q$ has a priori nothing to do with the properties of the vector field $\vec{\xi}$, which makes the method of little use. Our first task will be therefore to obtain an alternative (and completely general) expression for $L_{\vec{m}}Q$ in terms of $\vec{\xi}$, or more specifically, in terms of its deformation tensor $a_{\mu\nu}(\vec{\xi})$. We will devote

Section 3.3 to doing this. The result, given in Proposition 3.3.1, is thoroughly used in this chapter and also has independent interest.

With this expression at hand, we will be able to analyze under which conditions our geometrical procedure gives restrictions on $\vec{\xi}$. In Section 3.4 we will concentrate on the case where $L_{\vec{m}}Q$ has a sign everywhere on S. The main result of Section 3.4 will be given in Theorem 3.4.2, which holds for any vector field $\vec{\xi}$. This result will be then particularized to conformal Killing vectors (including homotheties and Killing vectors) in Corollary 3.4.3. Under the additional restriction that the homothety or the Killing vector is everywhere causal and future (or past) directed, strong restrictions on the geometry of the MOTS will be derived (Corollary 3.4.4). As a consequence, we will prove that in a plane wave spacetime any stable MOTS must be orthogonal to the direction of propagation of the wave. Marginally trapped surfaces will be also discussed in this section.

As an explicit application of the results on conformal Killing vectors, we will show, in Subsection 3.4.1, that stable MOTS cannot exist in any spacelike hypersurface in FLRW cosmological models provided the density μ and pressure p satisfy the inequalities $\mu \geq 0$, $\mu \geq 3p$ and $\mu + p \geq 0$. This includes, for instance, all classic models of matter and radiation dominated eras and also those models with accelerated expansion which satisfy the NEC. Subsection 3.4.2 will deal with one case where, in contrast with the standard situation, the geometric construction does in fact give sharper results than the elliptic theory. One of these results, together with Theorem 2.2.31 by Andersson and Metzger, will imply an interesting result (Theorem 3.4.10) for weakly outer trapped surfaces in stationary spacetimes.

In the case when $L_{\vec{m}}Q$ is not assumed to have a definite sign, the maximum principle loses its power. However, as we will discuss in Section 3.5, a result by Kriele and Hayward [77] will allow us to exploit our geometric construction again to obtain additional results. This will produce a theorem (Theorem 3.5.2) which holds for general vector fields $\vec{\xi}$ on any locally outermost MOTS. As in the previous section, we will particularize the result to conformal Killing vectors, and then to causal Killing vectors and homotheties which, in this case, will be allowed to change their time orientation on S

The results presented in this chapter have been published mainly in the papers [26], [27] and partly in [24] and [25].

3.2 Geometric procedure

Consider a spacelike hypersurface (Σ, g, K) which is embedded in a spacetime $(M, g^{(4)})$ with a vector field $\vec{\xi}$ defined on a neighbourhood of Σ. Assume that Σ possesses a barrier S_b with interior Ω_b and let $S \subset \Sigma$ be a bounding MOTS with respect to S_b (and therefore an *exterior region* of S in Ω_b can be properly defined). The idea we want to exploit consists in constructing under certain circumstances a new weakly outer trapped surface $S_\tau \subset \Omega_b$ which lies, at least partially, outside S. This fact will provide a contradiction in the case when S is the outermost bounding MOTS and will allow us to obtain restrictions on the vector $\vec{\xi}$ on S. As we will see below, this simple idea will allow us to obtain results also for stable, strictly stable and locally outermost MOTS, irrespectively of whether they are bounding or not, by using the theory of elliptic second order operators.

The geometric procedure to construct the new surface S_τ consists in moving S first along the integral lines of $\vec{\xi}$ a parametric amount τ. This gives a new surface S'_τ. Next, take the null normal $\vec{l}'_+(\tau)$ on this surface which coincides with the continuous deformation of the outer null normal $\vec{l}_+$ on S normalized to satisfy $l_+^\mu n_\mu = -1$ (where $\vec{n}$ denotes the unit vector normal to Σ and future directed) and consider the null hypersurface generated by null geodesics with tangent vector $\vec{l}'_+(\tau)$. This hypersurface is smooth close enough to S'_τ. Being null, its intersection with the spacelike hypersurface Σ is transversal and hence defines a smooth surface S_τ (for τ sufficiently small). By this construction, a point $\mathfrak{p}$ on S describes a curve in Σ when τ is varied. The tangent vector of this curve on S, denoted by $\vec{\nu}$, will define the variation vector generating the one-parameter family $\{S_\tau\}_{\tau\in I\subset\mathbb{R}}$ on a neighbourhood of S in Σ. Figure 3.1 gives a graphic representation of this construction.

Let us decompose the vector $\vec{\xi}$ into normal and tangential components with respect to Σ, as $\vec{\xi} = N\vec{n} + \vec{Y}$ (see Figure 3.2). On S we will further decompose $\vec{Y}$ in terms of a tangential component $\vec{Y}^{\parallel}$, and a normal component $(Y_i m^i)\vec{m}$, where $\vec{m}$ is the unit vector normal to S in Σ which points to the exterior of S in Σ. Therefore, $\vec{\xi}|_S = N_S\vec{n} + (Y_i m^i)\vec{m} + \vec{Y}^{\parallel}$, where N_S is the value of N on the surface. In order to study the variation vector $\vec{\nu}$, let us expand the embedding functions $\left\{x^\mu\left(y^A, \tau\right)\right\}$ of the surface S_τ (where $\left\{y^A\right\}$ are intrinsic coordinates of S) as

$$x^\mu\left(y^A,\tau\right) = x^\mu\left(y^A,0\right) + \xi^\mu\left(y^A,0\right)\tau + F(y^A) l'_+(\tau)^\mu\left(y^A\right)\tau + O(\tau^2), \qquad (3.2.1)$$

where $F(y^A)$ is a function to be adjusted. Since $\vec{\nu}$ defines the variation of S to first order, equation (3.2.1) implies that we only need to evaluate the vector $\vec{l}'_+(\tau)$

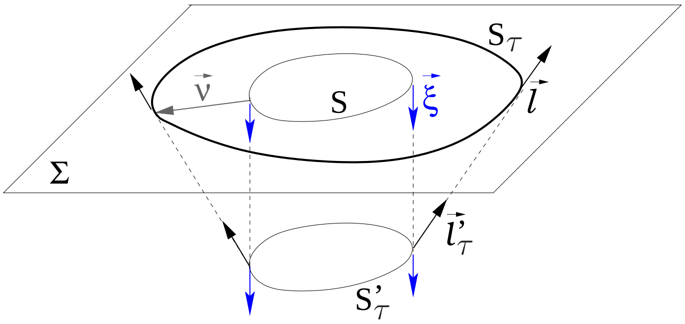


Figure 3.1: The figure represents how the new surface S_t is constructed from the original surface S. The intermediate surface S'_τ is obtained from S by dragging along $\vec{\xi}$ a parametric amount τ. Although $\vec{\xi}$ has been depicted as timelike here, this vector can be in fact of any causal character.

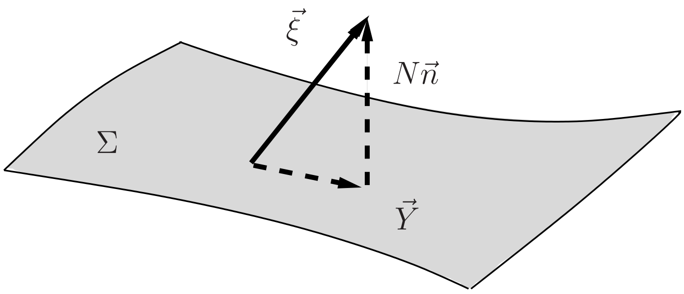


Figure 3.2: The vector $\vec{\xi}$ decomposed into normal $N\vec{n}$ and tangential $\vec{Y}$ components.

to zero order in τ, which obviously coincides with $\vec{l}_+$. It follows then that $\vec{\nu}$ is a linear combination (with functions) of $\vec{\xi}$ and $\vec{l}_+$. The amount we need to move S'_τ in order to go back to Σ can be determined by imposing $\vec{\nu}$ to be tangent to Σ. Since $\vec{\nu}(y^A) = \vec{\xi}(y^A) + F(y^A)\vec{l}_+(y^A)$, multiplication with $\vec{n}$ gives $0 = N_S + F$. Thus, $F = -N_S$ and $\vec{\nu} = \vec{\xi} - N_S\vec{l}_+$. Using the previous decomposition for $\vec{\xi}$ and $\vec{l}_+ = \vec{n} + \vec{m}$ we can rewrite $\vec{\nu} = Q\vec{m} + \vec{Y}^{\parallel}$, where

$$Q = (Y_i m^i) - N_S = \xi_\mu l_+^\mu \tag{3.2.2}$$

determines at first order the amount and sense to which a point $\mathfrak{p} \in S$ moves along the normal direction.

Let us consider for a moment the simplest case that $\vec{\xi}$ is a Killing vector. Suppose S is a MOTS which is bounding with respect to a barrier S_b with interior Ω_b. Since the null expansion does not change under an isometry, it follows that the surface S'_τ is also a bounding MOTS for the spacelike hypersurface obtained by moving Σ along the integral curves of $\vec{\xi}$ an amount τ. Moving back to Σ along the null hypersurface gives a contribution to $\theta^+[S_\tau]$ which is easily computed to be $\frac{d}{d\tau}\left[\hat{\varphi}^*_\tau(\theta^+[S_\tau])\right]\Big|_{\tau=0} = \frac{1}{2}N{\theta^+}^2[S] + NW\Big|_S$ which is the well-known Raychaudhuri equation (which has already appeared before in equation (2.2.13) for the particular case of MOTS), where $\hat{\varphi}_\tau : S \to S_\tau$ is the diffeomorphism defined by the geometrical construction above and W was defined in equation (2.2.14) and is non-negative provided the NEC holds. It implies that if $N_S < 0$ and $W \neq 0$ everywhere, then $\theta^+[S_\tau] < 0$ provided τ is positive and sufficiently small and the NEC holds. Therefore, S_τ is a bounding (provided τ is sufficiently small) weakly outer trapped surface which lies partially outside S if $Q > 0$ somewhere. This is impossible if S is an outermost bounding MOTS by Theorem 2.2.31 of Andersson and Metzger. Thus, the function Q must be non-positive everywhere on any outermost bounding MOTS S for which $N_S < 0$ and $W \neq 0$ everywhere.

Independently of whether $\vec{\xi}$ is a Killing vector or not, the more favorable case to obtain restrictions on the generator $\vec{\xi}$ on a given outermost bounding MOTS is when the newly constructed surface S_τ is bounding and weakly outer trapped. This is guaranteed for small enough τ when $\delta_{\vec{\nu}}\theta^+$ is strictly negative everywhere, because then this first order terms becomes dominant for small enough τ. Due to the fact that the tangential part of $\vec{\nu}$ does not affect the variation of θ^+ along $\vec{\nu}$ for a MOTS (c.f. (2.2.10)), it follows that $\delta_{\vec{\nu}}\theta^+ = L_{\vec{m}}Q$, where $L_{\vec{m}}$ is the stability operator for MOTS defined in (2.2.11). Since the vector $\vec{\nu} = Q\vec{m} + \vec{Y}^{\parallel}$ determines to first order the direction to which a point $\mathfrak{p} \in S$ moves, it is clear that $L_{\vec{m}}Q < 0$ everywhere and $Q > 0$ somewhere is impossible for an outermost bounding

MOTS. This is precisely the argument we have used above and is intuitively very clear. However, this geometric method does not provide the most powerful way of finding this type of restriction. Indeed, when the first order term $L_{\vec{m}}Q$ vanishes at some points, then higher order coefficients come necessarily into play, which makes the geometric argument of little use. It is remarkable that using the elliptic results described in Appendix B, most of these situations can be treated in a satisfactory way. Furthermore, since the elliptic methods only use infinitesimal information, there is no need to restrict oneself to outermost bounding MOTS, and the more general case of stable or strictly stable MOTS (not necessarily bounding) can be considered.

Unfortunately, the general expression of $L_{\vec{m}}Q$ given in equation (2.2.11) is not directly linked to the vector $\vec{\xi}$, which is clearly unsuitable for our aims. In the case of Killing vectors, the point of view of moving S along $\vec{\xi}$ and then back to Σ gives a simple method of calculating $L_{\vec{m}}Q$. For more general vectors, however, the motion along $\vec{\xi}$ will give a non-zero contribution to θ^+ which needs to be computed (for Killing vectors this term was known to be zero via a symmetry argument, not from a direct computation). In order to do this, it becomes necessary to have an alternative, and completely general, expression for $\delta_{\vec{\xi}}\theta^+$ directly in terms of the deformation tensor $a_{\mu\nu}(\vec{\xi})$ associated with $\vec{\xi}$. This is the aim of the following section.

3.3 Variation of the expansion and the metric deformation tensor

Let us derive an identity for $\delta_{\vec{\xi}}\theta^+$ in terms of $a_{\mu\nu}(\vec{\xi})$. This result will be important later on in this chapter, and may also be of independent interest. We derive this expression in full generality, without assuming S to be a MOTS and for the expansion $\theta_{\vec{\eta}}$ along any normal vector $\vec{\eta}$ of S (not necessarily a null normal) i.e.

$$\theta_{\vec{\eta}} \equiv H_\mu \eta^\mu,$$

where $\vec{H}$ denotes the mean curvature of S in M.

To do this calculation, we need to take derivatives of tensorial objects defined on each one of S'_τ. For a given point $\mathfrak{p} \in S$, these tensors live on different spaces, namely the tangent spaces of $\varphi_\tau(\mathfrak{p})$, where φ_τ is the one-parameter local group of diffeomorphisms generated by $\vec{\xi}$. In order to define the variation, we need to pull-back all these tensors to the point $\mathfrak{p}$ before doing the derivative. We will denote

the resulting derivative by $\mathscr{L}_{\vec{\xi}}$. In general, this operation is not the standard Lie derivative $\mathcal{L}_{\vec{\xi}}$ on tensors because it is applied to tensorial objects on each S'_τ which may not define tensor fields on M (e.g. when these surfaces intersect each other). Nevertheless, both derivatives do coincide when acting on spacetime tensor fields (e.g. the metric $g^{(4)}$) which will simplify the calculation considerably.

Notice in particular that the definition of $\theta_{\vec{\eta}}$ depends on the choice of $\vec{\eta}$ on each of the surfaces S'_τ. Thus $\delta_{\vec{\xi}}\theta_{\vec{\eta}} \equiv \mathscr{L}_{\vec{\xi}}\theta_{\vec{\eta}}\Big|_S$ will necessarily include a term of the form $\mathscr{L}_{\vec{\xi}}\eta_\alpha$ which is not uniquely defined (unless $\vec{\eta}$ can be uniquely defined on each S'_τ, which is usually not the case). Nevertheless, for the case of MOTS and when $\vec{\eta} = \vec{l}_+$ this a priori ambiguous term becomes determined, as we will see. The general expression for $\delta_{\vec{\xi}}\theta_{\vec{\eta}}$ is given in the following proposition.

Proposition 3.3.1 *Let S be a surface on a spacetime $(M, g^{(4)})$, $\vec{\xi}$ a vector field defined on M with deformation tensor $a_{\mu\nu}(\vec{\xi})$ and $\vec{\eta}$ a vector field normal to S and extend $\vec{\eta}$ to a smooth map $\vec{\eta} : (-\epsilon, \epsilon) \times S \to TM$ satisfying $\vec{\eta}(0, \mathfrak{p}) = \vec{\eta}(\mathfrak{p})$ and $\vec{\eta}(\tau, \mathfrak{p}) \in (T_{\varphi_\tau(\mathfrak{p})}S'_\tau)^\perp$ where φ_τ is the local group of diffeomorphisms generated by $\vec{\xi}$ and $S'_\tau = \varphi_\tau(S)$. Then, the variation along $\vec{\xi}$ of the expansion $\theta_{\vec{\eta}}$ on S reads*

$$\begin{aligned}\delta_{\vec{\xi}}\theta_{\vec{\eta}} &= H^\mu \mathscr{L}_{\vec{\xi}}\eta_\mu - a_{AB}(\vec{\xi})\Pi^{AB}_\mu \eta^\mu \\ &\quad + \gamma^{AB} e^\alpha_A e^\rho_B \eta^\nu \left[\frac{1}{2}\nabla_\nu a_{\alpha\rho}(\vec{\xi}) - \nabla_\alpha a_{\nu\rho}(\vec{\xi})\right]\Bigg|_S ,\end{aligned} \tag{3.3.1}$$

where $\vec{\Pi}_{AB}$ denotes the second fundamental form vector of S in M, and $a_{AB}(\vec{\xi}) \equiv e^\alpha_A e^\beta_B a_{\alpha\beta}(\vec{\xi})$, with $\{\vec{e}_A\}$ being a local basis for TS.

Proof. Since $\theta_{\vec{\eta}} = H_\mu \eta^\mu = \gamma^{AB}\Pi^\mu_{AB}\eta_\mu$, the variation we need to calculate involves three terms

$$\mathscr{L}_{\vec{\xi}}\theta_{\vec{\eta}} = \left(\mathscr{L}_{\vec{\xi}}\gamma^{AB}\right)\Pi^\mu_{AB}\eta_\mu + \gamma^{AB}\left(\mathscr{L}_{\vec{\xi}}\Pi^\mu_{AB}\right) + H^\mu\left(\mathscr{L}_{\vec{\xi}}\eta_\mu\right). \tag{3.3.2}$$

In order to do the calculation, we will choose $\varphi_{\tau\star}(\vec{e}_A)$ as the basis of tangent vectors at $\varphi_\tau(\mathfrak{p}) \in S'_\tau$ (we refer to $\varphi_{\tau\star}(\vec{e}_A)$ merely as $\vec{e}_A$ in the following to simplify the notation). This entails no loss of generality and implies $\mathscr{L}_{\vec{\xi}}\vec{e}_A = 0$, which makes the calculation simpler. Our aim is to express each term of (3.3.2) in terms of $a_{\mu\nu}(\vec{\xi})$. For the first term, we need to calculate $\mathscr{L}_{\vec{\xi}}\gamma^{AB}$. We start with $\mathscr{L}_{\vec{\xi}}\gamma_{AB} = \mathscr{L}_{\vec{\xi}}\left(g^{(4)}(\vec{e}_A, \vec{e}_B)\right) = \left(\mathscr{L}_{\vec{\xi}}g\right)(\vec{e}_A, \vec{e}_B) = \left(\mathcal{L}_{\vec{\xi}}g\right)(\vec{e}_A, \vec{e}_B) = a_{\mu\nu}(\vec{\xi})e^\mu_A e^\nu_B \equiv a_{AB}(\vec{\xi})$, which immediately implies $\mathscr{L}_{\vec{\xi}}\gamma^{AB} = -a_{CD}(\vec{\xi})\gamma^{AC}\gamma^{BD}$, so that the first term in (3.3.2) becomes

$$\mathscr{L}_{\vec{\xi}}\gamma^{AB}\Pi^\mu_{AB}\eta_\mu = -a_{AB}(\vec{\xi})\Pi^{AB}_\mu\eta^\mu. \tag{3.3.3}$$

The second term $\gamma^{AB}(\mathscr{L}_{\vec{\xi}}\Pi^{\mu}_{AB})\eta_\mu$ is more complicated. It is useful to introduce the projector to the normal space of S, $h^{\mu}_{\nu} \equiv \delta^{\mu}_{\nu} - g^{(4)}_{\nu\beta} e^{\mu}_{A} e^{\beta}_{B} \gamma^{AB}$. From the previous considerations, it follows that $\mathscr{L}_{\vec{\xi}} h^{\mu}_{\nu} = e^{\mu}_{A} e^{\beta}_{B} (a^{AB}(\vec{\xi}) g^{(4)}_{\nu\beta} - \gamma^{AB} a_{\nu\beta}(\vec{\xi}))$, which implies $\left(\mathscr{L}_{\vec{\xi}} h^{\mu}_{\nu}\right)\eta_\mu = 0$ and hence

$$\mathscr{L}_{\vec{\xi}}(\Pi^{\mu}_{AB})\eta_\mu = -\mathscr{L}_{\vec{\xi}}\left(h^{\mu}_{\nu} e^{\alpha}_{A} \nabla_\alpha e^{\nu}_{B}\right)\eta_\mu = -\eta_\nu \mathscr{L}_{\vec{\xi}}\left(e^{\alpha}_{A}\nabla_\alpha e^{\nu}_{B}\right). \tag{3.3.4}$$

Therefore we only need to evaluate $\mathscr{L}_{\vec{\xi}}(e^{\alpha}_{A}\nabla_\alpha e^{\nu}_{B})$. It is well-known that for an arbitrary vector field $\vec{v}$, $\mathcal{L}_{\vec{\xi}}\nabla_\alpha v^\nu - \nabla_\alpha \mathcal{L}_{\vec{\xi}} v^\nu = v^\rho \nabla_\alpha \nabla_\rho \xi^\nu + {R^{(4)}}^{\nu}{}_{\rho\sigma\alpha} v^\rho \xi^\sigma$. However, this expression is not directly applicable to the variational derivative we are calculating and we need the following closely related lemma.

Lemma 3.3.2

$$\mathscr{L}_{\vec{\xi}}\left(e^{\alpha}_{A}\nabla_\alpha e^{\nu}_{B}\right) = e^{\alpha}_{A} e^{\rho}_{B} \nabla_\alpha \nabla_\rho \xi^\nu + {R^{(4)}}^{\nu}{}_{\rho\sigma\alpha} e^{\alpha}_{A} e^{\rho}_{B} \xi^\sigma. \tag{3.3.5}$$

Proof of Lemma 3.3.2. Choose coordinates y^A on S and extend them as constants along $\vec{\xi}$. This gives coordinates on each one of S'_τ. Define $e^{\alpha}_{A} = \frac{\partial x^\alpha}{\partial y^A}$, where $x^\mu(y^A,\tau)$ are the embedding functions of S'_τ in M in spacetime coordinates x^μ. The map $\varphi_{-\tau} : M \to M$ relates every point $\mathfrak{p} \in S_\tau$ with coordinates $\{x^\alpha\}$ to a point $\varphi_{-\tau}(\mathfrak{p}) \in S$ with coordinates $\{\varphi^{\alpha}_{-\tau}(x^\beta)\}$. By definition, $\mathscr{L}_{\vec{\xi}}(e^{\mu}_{A}\nabla_\mu e^{\nu}_{B}) \equiv \frac{d}{d\tau}\left((\varphi_{-\tau})_*(e^{\mu}_{A}\nabla_\mu e^{\nu}_{B})\right)$. Using that $\frac{\partial \varphi^{\alpha}_{-\tau}(x^\beta)}{\partial\tau} = -\xi^\alpha$, it is immediate to obtain

$$\begin{aligned}
\frac{d}{d\tau}\left((\varphi_{-\tau})_*(e^{\mu}_{A}\nabla_\mu e^{\nu}_{B})\right)\Big|_{\tau=0} &= \frac{d}{d\tau}\left[(e^{\mu}_{A}\nabla_\mu e^{\alpha}_{B})\frac{\partial \varphi^{\nu}_{-\tau}}{\partial x^\alpha}\right]\Bigg|_{\tau=0}\\
&= \frac{\partial}{\partial\tau}\left(e^{\mu}_{A}\nabla_\mu e^{\nu}_{B}(y^C,\tau)\right) - \partial_\alpha \xi^\nu e^{\mu}_{A}\nabla_\mu e^{\alpha}_{B}\\
&= \frac{\partial}{\partial\tau}\left[\frac{\partial^2 x^\nu}{\partial y^A \partial y^B} + \Gamma^{\nu}_{\alpha\rho}\frac{\partial x^\alpha}{\partial y^A}\frac{\partial x^\rho}{\partial y^B}\right] - \partial_\mu \xi^\nu\left[\frac{\partial^2 x^\mu}{\partial y^A \partial y^B} + \Gamma^{\mu}_{\alpha\rho}\frac{\partial x^\alpha}{\partial y^A}\frac{\partial x^\rho}{\partial y^B}\right].
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&e^{\alpha}_{A} e^{\rho}_{B}\nabla_\alpha\nabla_\rho \xi^\nu + {R^{(4)}}^{\nu}{}_{\rho\sigma\alpha} e^{\alpha}_{A} e^{\rho}_{B}\xi^\sigma\\
&\quad= \frac{\partial x^\alpha}{\partial y^A}\frac{\partial x^\rho}{\partial y^B}\left[\partial_\alpha\partial_\rho\xi^\nu + \Gamma^{\nu}_{\mu\rho}\partial_\alpha\xi^\mu + \Gamma^{\nu}_{\mu\alpha}\partial_\rho\xi^\mu - \Gamma^{\mu}_{\alpha\rho}\partial_\mu\xi^\nu + \xi^\sigma\partial_\sigma\Gamma^{\nu}_{\alpha\rho}\right]\\
&\quad= \frac{\partial^3 x^\nu}{\partial\tau\partial y^A\partial y^B} - \frac{\partial^2 x^\rho}{\partial y^A\partial y^B}\partial_\rho\xi^\nu + \frac{\partial x^\rho}{\partial y^B}\Gamma^{\nu}_{\mu\rho}\partial_\tau\left(\frac{\partial x^\mu}{\partial y^A}\right) + \frac{\partial x^\alpha}{\partial y^A}\Gamma^{\nu}_{\mu\alpha}\partial_\tau\left(\frac{\partial x^\mu}{\partial y^B}\right)\\
&\qquad+ \frac{\partial x^\alpha}{\partial y^A}\frac{\partial x^\rho}{\partial y^B}\left[\partial_\tau\Gamma^{\nu}_{\alpha\rho} - \Gamma^{\mu}_{\alpha\rho}\partial_\mu\xi^\nu\right]\\
&\quad= \frac{\partial}{\partial\tau}\left[\frac{\partial^2 x^\nu}{\partial y^A\partial y^B} + \Gamma^{\nu}_{\alpha\rho}\frac{\partial x^\alpha}{\partial y^A}\frac{\partial x^\rho}{\partial y^B}\right] - \partial_\mu\xi^\nu\left[\frac{\partial^2 x^\mu}{\partial y^A\partial y^B} + \Gamma^{\mu}_{\alpha\rho}\frac{\partial x^\alpha}{\partial y^A}\frac{\partial x^\rho}{\partial y^B}\right],
\end{aligned}$$

where we have used

$$R^{(4)\nu}{}_{\rho\sigma\alpha} = \partial_\sigma \Gamma^\nu_{\rho\alpha} - \partial_\alpha \Gamma^\nu_{\rho\sigma} + \Gamma^\nu_{\gamma\sigma}\Gamma^\gamma_{\rho\alpha} - \Gamma^\nu_{\gamma\alpha}\Gamma^\gamma_{\rho\sigma},$$

in the first equality and $\xi^\mu = \frac{\partial x^\mu(y^A,\tau)}{\partial \tau}$ in the second one. This proves the lemma. ■

We can now continue with the proof of Proposition 3.3.1. It only remains to express the quantity $\nabla_\alpha \nabla_\rho \xi^\nu + R^{(4)\nu}{}_{\rho\sigma\alpha}\xi^\sigma$ in terms of $a_{\mu\nu}(\vec{\xi})$. To that end, we take a derivative of $\nabla_\nu \xi_\rho + \nabla_\rho \xi_\nu = a_{\nu\rho}(\vec{\xi})$ to get

$$\nabla_\alpha \nabla_\nu \xi_\rho + \nabla_\alpha \nabla_\rho \xi_\nu = \nabla_\alpha a_{\nu\rho}(\vec{\xi}),$$

and use the Ricci identity $\nabla_\alpha \nabla_\nu \xi_\rho - \nabla_\nu \nabla_\alpha \xi_\rho = -R^{(4)}{}_{\sigma\rho\alpha\nu}\xi^\sigma$ to obtain

$$\nabla_\nu \nabla_\alpha \xi_\rho + \nabla_\alpha \nabla_\rho \xi_\nu = R^{(4)}{}_{\sigma\rho\alpha\nu}\xi^\sigma + \nabla_\alpha a_{\nu\rho}(\vec{\xi}).$$

Now, write the three equations obtained from this one by cyclic permutation of the three indices. Adding two of them and subtracting the third one we find

$$\begin{aligned} \nabla_\alpha \nabla_\rho \xi_\nu &= \frac{1}{2}(R^{(4)}{}_{\sigma\rho\alpha\nu} + R^{(4)}{}_{\sigma\nu\rho\alpha} - R^{(4)}{}_{\sigma\alpha\nu\rho})\xi^\sigma \\ &\quad + \frac{1}{2}\left[\nabla_\alpha a_{\nu\rho}(\vec{\xi}) + \nabla_\rho a_{\alpha\nu}(\vec{\xi}) - \nabla_\nu a_{\alpha\rho}(\vec{\xi})\right]. \end{aligned}$$

which, after using the first Bianchi identity $R^{(4)}_{\sigma\rho\alpha\nu} + R^{(4)}_{\sigma\nu\rho\alpha} + R^{(4)}_{\sigma\alpha\nu\rho} = 0$, leads to

$$\nabla_\alpha \nabla_\rho \xi_\nu = R^{(4)}{}_{\sigma\alpha\rho\nu}\xi^\sigma + \frac{1}{2}\left[\nabla_\alpha a_{\nu\rho}(\vec{\xi}) + \nabla_\rho a_{\alpha\nu}(\vec{\xi}) - \nabla_\nu a_{\alpha\rho}(\vec{\xi})\right].$$

Substituting (3.3.5) and this expression into (3.3.4) yields

$$\gamma^{AB}\mathscr{L}_{\vec{\xi}}\Pi^\mu_{AB}\eta_\mu = \gamma^{AB}e^\alpha_A e^\rho_B \eta^\nu \left[\frac{1}{2}\nabla_\nu a_{\alpha\rho}(\vec{\xi}) - \nabla_\alpha a_{\nu\rho}(\vec{\xi})\right]. \tag{3.3.6}$$

Inserting (3.3.3) and (3.3.6) into equation (3.3.2) proves the proposition. ■

We can now particularize to the outer null expansion in a MOTS.

Corollary 3.3.3 *If S is a MOTS then*

$$\begin{aligned} \delta_{\vec{\xi}}\theta^+ &= -\frac{1}{4}\theta^- a_{\mu\nu}(\vec{\xi})l^\mu_+ l^\nu_+ - a_{AB}(\vec{\xi})\Pi^{AB}_\mu l^\mu_+ \\ &\quad + \gamma^{AB}e^\alpha_A e^\rho_B l^\nu_+ \left[\frac{1}{2}\nabla_\nu a_{\alpha\rho}(\vec{\xi}) - \nabla_\alpha a_{\nu\rho}(\vec{\xi})\right]\bigg|_S. \end{aligned} \tag{3.3.7}$$

Proof. The normal vector $\vec{l}'_+(\tau)$ defined on each of the surfaces S'_τ is null. Therefore, using $\mathscr{L}_{\vec{\xi}}\, g^{(4)\mu\nu} = \mathcal{L}_{\vec{\xi}}\, g^{(4)\mu\nu} = -a^{\mu\nu}(\vec{\xi})$,

$$0 = \mathscr{L}_{\vec{\xi}} \left(l'_{+\mu}(\tau) l'_{+\nu}(\tau) g^{(4)\mu\nu} \right) = 2 l^\mu_+ \mathscr{L}_{\vec{\xi}}\, l'_{+\mu}(\tau) - a_{\mu\nu}(\vec{\xi}) l^\mu_+ l^\nu_+. \tag{3.3.8}$$

Since, on a MOTS $\vec{H} = -\frac{1}{2}\theta^- \vec{l}_+$, it follows $H^\mu \mathscr{L}_{\vec{\xi}}\, l'_{+\mu}(\tau) = -\frac{1}{2}\theta^- l^\mu_+ \mathscr{L}_{\vec{\xi}}\, l'_{+\mu}(\tau) = -\frac{1}{4}\theta^- a_{\mu\nu}(\vec{\xi}) l^\mu_+ l^\nu_+$, and the corollary follows from (3.3.1). ■

Remark. Formula (3.3.7) holds in general for arbitrary surfaces S at any point where $\theta^+ = 0$. □

3.4 Results provided $L_{\vec{m}} Q$ has a sign on S

In this section we will give several results provided $L_{\vec{m}} Q$ has a definite sign on S. In this case, a direct application of Lemma B.6 for a MOTS S with stability operator $L_{\vec{m}}$ leads to the following result.

Lemma 3.4.1 *Let S be a stable MOTS on a spacelike hypersurface Σ. If $L_{\vec{m}} Q|_S \leq 0$ (resp. $L_{\vec{m}} Q|_S \geq 0$) and not identically zero, then $Q|_S < 0$ (resp. $Q|_S > 0$).*

Furthermore, if S is strictly stable and $L_{\vec{m}} Q|_S \leq 0$ (resp. $L_{\vec{m}} Q|_S \geq 0$) then $Q|_S \leq 0$ (resp. $Q|_S \geq 0$) and it vanishes at one point only if it vanishes everywhere on S.

The general idea then is to combine Lemma 3.4.1 with the general calculation for the variation of θ^+ obtained in the previous section to get restrictions on special types of generators $\vec{\xi}$ on a stable or strictly stable MOTS. Our first result is fully general in the sense that it is valid for any generator $\vec{\xi}$.

Theorem 3.4.2 *Let S be a stable MOTS on a spacelike hypersurface Σ and $\vec{\xi}$ a vector field on S with deformation tensor $a_{\mu\nu}(\vec{\xi})$. With the notation above, define*

$$\begin{aligned} Z \;\; = \;\; & -\frac{1}{4}\theta^- a_{\mu\nu}(\vec{\xi}) l^\mu_+ l^\nu_+ - a_{AB}(\vec{\xi}) \Pi^{AB}_\mu l^\mu_+ \\ & \left. + \gamma^{AB} e^\alpha_A e^\rho_B l^\nu_+ \left[\frac{1}{2} \nabla_\nu a_{\alpha\rho}(\vec{\xi}) - \nabla_\alpha a_{\nu\rho}(\vec{\xi}) \right] + NW \right|_S , \end{aligned} \tag{3.4.1}$$

where $W = \Pi^\mu_{AB} \Pi^{\nu AB} l_{+\mu} l_{+\nu} + G_{\mu\nu} l^\mu_+ l^\nu_+$, and assume $Z \leq 0$ everywhere on S.

(i) If $Z \neq 0$ somewhere, then $\xi_\mu l^\mu_+ < 0$ everywhere.

(ii) If S is strictly stable, then $\xi_\mu l_+^\mu \leq 0$ everywhere and vanishes at one point only if it vanishes everywhere.

Proof. Consider the first variation of S defined by the vector $\vec{\nu} = \vec{\xi} - N_S \vec{l}_+ = Q\vec{m} + \vec{Y}^{\parallel}$. From equation (2.2.10) and Definition 2.2.20 we have $\delta_{\vec{\nu}}\theta^+ = L_{\vec{m}}Q$. On the other hand, linearity of this variation under addition gives $\delta_{\vec{\nu}}\theta^+ = \delta_{\vec{\xi}}\theta^+ - \delta_{N_S\vec{l}_+}\theta^+$. The Raychaudhuri equation for MOTS establishes that $\delta_{N_S\vec{l}_+}\theta^+ = -N_S W$ (see (2.2.13) and (2.2.14)) and the identity (3.3.7) gives $L_{\vec{m}}Q = Z$. Since $Q = \xi_\mu l_+^\mu$, the result follows directly from Lemma 3.4.1. ■

Remark. The theorem also holds if all the inequalities are reversed. This follows directly by replacing $\vec{\xi} \to -\vec{\xi}$. □

This theorem gives information about the relative position between the generator $\vec{\xi}$ and the outer null normal $\vec{l}_+$ and has, in principle, many potential consequences. Specific applications require considering spacetimes having special vector fields for which sufficient information about its deformation tensor is available. Once such a vector is known to exist, the result above can be used either to restrict the form of $\vec{\xi}$ in stable or strictly stable MOTS or, alternatively, to restrict the regions of the spacetime where such MOTS are allowed to be present.

Since conformal vector fields (and homotheties and isometries as particular cases) have very special deformation tensors, the theorem above gives interesting information for spacetimes admitting such symmetries.

Corollary 3.4.3 *Let S be a stable MOTS in a hypersurface Σ of a spacetime $(M, g^{(4)})$ which admits a conformal Killing vector $\vec{\xi}$, $\mathcal{L}_{\vec{\xi}} g^{(4)}_{\mu\nu} = 2\phi g^{(4)}_{\mu\nu}$ (including homotheties $\phi = C$, and isometries $\phi = 0$).*

(i) If $2\vec{l}_+(\phi) + NW|_S \leq 0$ and not identically zero, then $\xi_\mu l_+^\mu|_S < 0$.

(ii) If S is strictly stable and $2\vec{l}_+(\phi) + NW|_S \leq 0$ then $\xi_\mu l_+^\mu|_S \leq 0$ and vanishes at one point only if it vanishes everywhere

Remark 1. As before, the theorem is still true if all inequalities are reversed. □

Remark 2. In the case of homotheties and Killing vectors, the condition of the theorem demands that $N_S W \leq 0$. Under the NEC, this holds provided

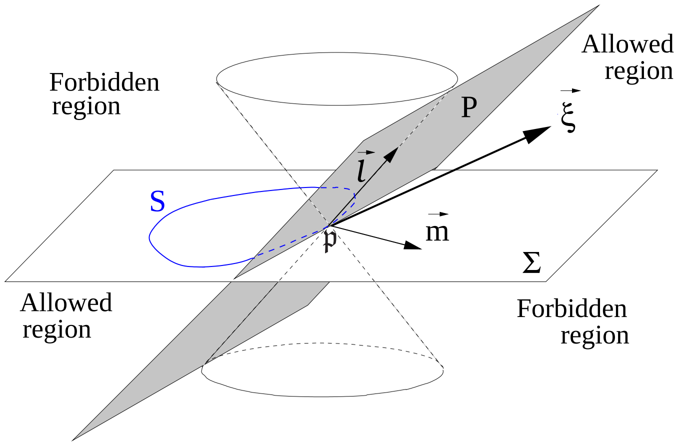


Figure 3.3: The planes $T_{\mathfrak{p}}\Sigma$ and $P \equiv T_{\mathfrak{p}}S \oplus \text{span}\{\vec{l}_+ |_{\mathfrak{p}}\}$ divide the tangent space $T_{\mathfrak{p}}M$ in four regions. By Corollary 3.4.3, if S is strictly stable and $\vec{\xi}$ is a Killing vector or a homothety in a spacetime satisfying the NEC which points above Σ everywhere, then $\vec{\xi}$ cannot enter into the forbidden region at any point (and similarly, if $\vec{\xi}$ points below Σ everywhere). The allowed region includes the plane P. However, if there is a point with $W \neq 0$ where $\vec{\xi}$ is not tangent to Σ, then the result is also valid for stable MOTS with P belonging to the forbidden region.

$N_S \leq 0$, i.e. when $\vec{\xi}$ points below Σ everywhere on S (where the term "below" includes also the tangential directions). For strictly stable S, the conclusion of the theorem is that the homothety or the Killing vector must lie above the null hyperplane defined by the tangent space of S and the outer null normal $\vec{l}_+$ at each point $\mathfrak{p} \in S$. If the MOTS is only assumed to be stable, then the theorem requires the extra condition that $\vec{\xi}$ points strictly below Σ at some point with $W \neq 0$. In this case, the conclusion is stronger and forces $\vec{\xi}$ to lie strictly above the null hyperplane everywhere. By changing the orientation of $\vec{\xi}$, it is clear that similar restrictions arise when $\vec{\xi}$ is assumed to point *above* Σ. Figure 3.3 summarizes the allowed and forbidden regions for $\vec{\xi}$ in this case. □

Proof. We only need to show that $Z = 2\vec{l}_+(\phi) + NW|_S$ for conformal Killing vectors. This follows at once from (3.4.1) and $a_{\mu\nu}(\vec{\xi}) = 2\phi g^{(4)}_{\mu\nu}$ after using orthogonality of $\vec{e}_A$ and $\vec{l}_+$. Notice in particular that Z is the same for isometries and for homotheties. ■

This corollary has an interesting consequence in spacetime regions where there exists a Killing vector or a homothety $\vec{\xi}$ which is causal everywhere.

Corollary 3.4.4 *Let a spacetime $(M, g^{(4)})$ satisfying the NEC admit a causal Killing vector or homothety $\vec{\xi}$ which is future (or past) directed everywhere on a stable MOTS $S \subset \Sigma$. Then,*

(i) The second fundamental form Π^{+}_{AB} along $\vec{l}_+$ (i.e. $\Pi^{+}_{AB} \equiv \Pi^{\mu}_{AB} l_{+\mu}$) and $G^{(4)}_{\mu\nu} l^{\mu}_{+} l^{\nu}_{+}$ vanish identically on every point $\mathfrak{p} \in S$ where $\vec{\xi}|_{\mathfrak{p}} \neq 0$.

(ii) If S is strictly stable, then $\vec{\xi} \propto \vec{l}_+$ everywhere.

Remark. If we assume that there exists an open neighbourhood of S in M where the Killing vector or homothety $\vec{\xi}$ is causal and future (or past) directed everywhere then the conclusion (i) can be generalized to say that Π^{+}_{AB} and $G_{\mu\nu} l^{\mu}_{+} l^{\nu}_{+}$ vanish identically on S. The reason is that such a $\vec{\xi}$ cannot vanish anywhere in this neighbourhood (and consequently neither on S). For Killing vectors this result is proven in Lemma 3.2 in [12][1]. A simple generalization shows that the same holds for homotheties, as follows. Suppose that $\vec{\xi}\,|_{\mathfrak{p} \in S} = 0$. Take a timelike affine-parametrized geodesic γ passing through $\mathfrak{p}$ with future directed unit tangent vector $\vec{v}$. A simple computation gives that, if $\vec{\xi}$ is a homothety with constant C, $v^{\mu} \nabla_{\mu} (\xi_{\nu} v^{\nu}) = -C$. Supposing $C > 0$, this implies that the causal vector $\vec{\xi}$ is future directed on the future of $\mathfrak{p}$ and past directed on the past of $\mathfrak{p}$ contradicting the fact that $\vec{\xi}$ is future (past) directed everywhere on a neighbourhood of S in M. A similar argument works if $C < 0$.

Point (ii) can be generalized to locally outermost MOTS using a finite construction. We will prove this in Theorem 3.4.9 below. □

Proof. We can assume, after reversing the sign of $\vec{\xi}$ if necessary, that $\vec{\xi}$ is past directed, i.e. $N_S \leq 0$.

Under the NEC, W is the sum of two non-negative terms, so in order to prove (i) we only need to show that $W = 0$ on points where $\vec{\xi} \neq 0$, i.e. at points where $N_S < 0$. Assume, on the contrary, that $W \neq 0$ and $N_S < 0$ happen simultaneously at a point $\mathfrak{p} \in S$. It follows that $N_S W \leq 0$ everywhere and non-zero at $\mathfrak{p}$. Thus, we can apply statement (i) of Corollary 3.4.3 to conclude $Q < 0$ everywhere. Hence $N_S Q \geq 0$ and not identically zero on S. Recalling the decomposition $\vec{\xi} = N_S \vec{l}_+ + Q\vec{m} + \vec{Y}^{\parallel}$, the squared norm of this vector is

$$\xi_{\mu}\xi^{\mu} = 2N_S Q + Q^2 + {Y^{\parallel}}_{\mu} {Y^{\parallel}}^{\mu}. \qquad (3.4.2)$$

This is the sum of non-negative terms, the first one not identically zero. This contradicts the condition of $\vec{\xi}$ being causal.

[1]We thank Miguel Sánchez Caja for pointing this out.

To prove the second statement, we notice that point (ii) in Corollary 3.4.3 implies $Q \leq 0$, and hence $N_S Q \geq 0$. The only way (3.4.2) can be negative or zero is if $Q = 0$ and $\vec{Y}^{\parallel} = 0$, i.e. $\vec{\xi} \propto \vec{l}_+$. ■

This corollary extends Theorem 2 in [82] to the case of stable MOTS and implies, for instance, that any strictly stable MOTS in a plane wave spacetime (which by definition admits a null and nowhere zero Killing vector field $\vec{\xi}$) must be aligned with the direction of propagation of the wave (in the sense that $\vec{\xi}$ must be one of the null normals to the surface). It also implies that any spacetime admitting a nowhere zero and causal Killing vector (or homothety) whose energy-momentum tensor satisfies the DEC and does not admit a null eigenvector cannot contain any stable MOTS. This is because $G^{(4)}_{\mu\nu} l_+^\mu l_+^\nu = 0$ and the DEC implies $G^{(4)}_{\mu\nu} l_+^\mu \propto l_\nu$ and $G^{(4)}_{\mu\nu}$ would have a null eigenvector. For perfect fluids this result holds even without the DEC provided $\mu + p \neq 0$ (this is because in this case $G^{(4)}_{\mu\nu} l_+^\mu l_+^\nu = (\mu + p)(l_+^\mu u_\mu)^2 \neq 0$ – where μ is the density, p the pressure and $\vec{u}$ is the 4-velocity of the fluid–).

The results above hold for stable or strictly stable MOTS. Among such surfaces, marginally trapped surfaces are of special interest. Our next result restricts (and in some cases forbids) the existence of such surfaces in spacetimes admitting Killing vectors, homotheties or conformal Killings.

Theorem 3.4.5 *Let S be a stable MOTS in a spacelike hypersurface Σ of a spacetime $(M, g^{(4)})$ which satisfies the NEC and admits a conformal Killing vector $\vec{\xi}$ with conformal factor $\phi \geq 0$ (including homotheties with $C \geq 0$ and Killing vectors). Suppose furthermore that either (i) $(2\vec{l}_+(\phi) + NW)|_S \not\equiv 0$ or (ii) S is strictly stable and $\xi_\mu l_+^\mu|_S \not\equiv 0$. Then the following holds.*

(a) If $2\vec{l}_+(\phi) + NW|_S \leq 0$ then S cannot be a marginally trapped surface, unless $\vec{H} \equiv 0$. The latter case is excluded if $\phi|_S \not\equiv 0$.

(b) If $2\vec{l}_+(\phi) + NW|_S \geq 0$ then S cannot be a past marginally trapped surface, unless $\vec{H} \equiv 0$. The latter case is excluded if $\phi|_S \not\equiv 0$.

Remark. The statement obtained from this one by reversing all the inequalities is also true. This is a direct consequence of the freedom in changing $\vec{\xi} \rightarrow -\vec{\xi}$. □

Proof. We will only prove case (a). The argument for case (b) is similar. The idea is taken from [82] and consists of performing a variation of S along the conformal Killing vector and evaluating the change of area in order to get

a contradiction if S is marginally trapped. The difference is that here we do not make any a priori assumption on the causal character for $\vec{\xi}$. Corollary 3.4.3 provides us with sufficient information for the argument to go through.

The first variation of area (2.2.3) gives

$$\delta_{\vec{\xi}}|S| = -\frac{1}{2}\int_S \theta^- \xi_\mu l_+^\mu \eta_S, \tag{3.4.3}$$

where we have used $\vec{H} = -\frac{1}{2}\theta^- \vec{l}_+$. Now, since $2\vec{l}_+(\phi) + NW|_S \leq 0$, and furthermore either hypothesis (i) or (ii) holds, Corollary 3.4.3 implies that $\xi_\mu l_+^\mu|_S < 0$.

On the other hand, $\vec{\xi}$ being a conformal Killing vector, the induced metric on S'_τ is related to the metric on S by conformal rescaling. A simple computation gives $\delta_{\vec{\xi}}\eta_S = \frac{1}{2}\gamma^{AB}(\mathcal{L}_{\vec{\xi}}g)(\vec{e}_A, \vec{e}_B)\eta_S$ (see e.g. [82]), which for the particular case of conformal Killing vectors gives the following.

$$\delta_{\vec{\xi}}|S| = 2\int_S \phi\eta_S, \tag{3.4.4}$$

This quantity is non-negative due to $\phi \geq 0$ and not identically zero if $\phi \neq 0$ somewhere. Combining (3.4.3) and (3.4.4) we conclude that if $\theta^- \leq 0$ (i.e. S is marginally trapped) then necessarily θ^- vanishes identically (and so does $\vec{H}$). Furthermore, if $\phi|_S$ is non-zero somewhere, then θ^- must necessarily be positive somewhere, and S cannot be marginally trapped. ■

3.4.1 An application: No stable MOTS in Friedmann-Lemaître-Robertson-Walker spacetimes

In this subsection we apply Corollary 3.4.3 to show that a large subclass of Friedmann-Lemaître-Robertson-Walker (FLRW) spacetimes do not admit stable MOTS on *any* spacelike hypersurface. Obtaining this type of results for metric spheres only requires a straightforward calculation, and is therefore simple. The power of the method is that it provides a general result involving no assumption on the geometry of the MOTS or on the spacelike hypersurface where it is embedded. The only requirement is that the scale factor and its time derivative satisfy certain inequalities. This includes, for instance all FLRW cosmological models satisfying the NEC with accelerated expansion, as we shall see in Corollary 3.4.7 below.

Recall that the FLRW metric is

$$g^{(4)}_{FLRW} = -dt^2 + a^2(t)\left[dr^2 + \chi^2(r;k)d\Omega^2\right],$$

where $a(t) > 0$ is the scale factor and $\chi(r;k) = \{\sin r, r, \sinh r\}$ for $k = \{1, 0, -1\}$, respectively. The Einstein tensor of this metric is of perfect fluid type and reads

$$G^{(4)}_{\mu\nu} = (\mu + p)u_\mu u_\nu + p g^{(4)}_{\mu\nu}, \quad \vec{u} = \partial_t, \quad \mu = \frac{3(\dot{a}^2(t) + k)}{a^2(t)}, \tag{3.4.5}$$

$$\mu + p = 2\left(\frac{\dot{a}^2(t) + k}{a^2(t)} - \frac{\ddot{a}(t)}{a(t)}\right) \tag{3.4.6}$$

where dot stands for derivative with respect to t.

Theorem 3.4.6 *There exists no stable MOTS in any spacelike hypersurface of a FLRW spacetime* $(M, g^{(4)}_{FLRW})$ *satisfying*

$$\frac{\dot{a}^2(t) + k}{a(t)} \geq 0, \quad -\frac{\dot{a}^2(t) + k}{a(t)} \leq \ddot{a}(t) \leq \frac{\dot{a}^2(t) + k}{a(t)}. \tag{3.4.7}$$

Remark. In terms of the energy-momentum contents of the spacetime, these three conditions read, respectively, $\mu \geq 0$, $\mu \geq 3p$ and $\mu + p \geq 0$. As an example, in the absence of a cosmological constant they are satisfied as soon as the weak energy condition is imposed and the pressure is not too large (e.g. for the matter and radiation dominated eras). The class of FLRW satisfying (3.4.7) is clearly very large (c.f. Corollary 3.4.7 below). We also remark that Theorem 3.4.6 agrees with the fact that the causal character of the hypersurface which separates the trapped from the non-trapped *spheres* in FLRW spacetimes depends precisely on the quantity $\mu^2(\mu + p)(\mu - 3p)$ (c.f. [105]). □

Proof. The FLRW spacetime admits a conformal Killing vector $\vec{\xi} = a(t)\vec{u}$ with conformal factor $\phi = \dot{a}(t)$. Since this vector is timelike and future directed, it follows that $\xi_\mu l_+^\mu|_S < 0$ for any spacelike surface S embedded in a spacelike hypersurface Σ. If we can show that $2\vec{l}_+(\phi) + NW|_S \geq 0$, and non-identically zero for any S, then the sign reversed of point (i) in Corollary 3.4.3 implies that S cannot be a stable MOTS, thus proving the result. The proof therefore relies on finding conditions on the scale factor which imply the validity of this inequality on any S. First of all, we notice that the second fundamental form Π^+_{AB} can be made as small as desired on a suitably chosen S. Thus, recalling that $W = \Pi^+{}_{AB}\Pi^{+AB} + G^{(4)}_{\mu\nu} l_+^\mu l_+^\nu$, it is clear that the inequality that needs to be satisfied is

$$2\vec{l}_+(\phi) + N G^{(4)}_{\mu\nu} l_+^\mu l_+^\nu \Big|_S \geq 0, \tag{3.4.8}$$

and positive somewhere. In order to evaluate this expression recall that $\vec{u} = a^{-1}\vec{\xi} = a(t)^{-1}N\vec{n} + a(t)^{-1}\vec{Y}$. Let us write $\vec{Y} = Y\vec{e}$, where $\vec{e}$ is unit and let α be

the hyperbolic angle of $\vec{u}$ in the basis $\{\vec{n}, \vec{e}\}$, i.e. $\vec{u} = \cosh\alpha\,\vec{n} + \sinh\alpha\,\vec{e}$. It follows immediately that $N = a(t)\cosh\alpha$ and $Y = a(t)\sinh\alpha$. Furthermore, multiplying $\vec{u}$ by the normal vector to the surface S in Σ we find $u_\mu m^\mu = \cos\varphi\sinh\alpha$, where φ is the angle between $\vec{m}$ and $\vec{e}$. With this notation, let us calculate the null vector $\vec{l}_+$. Writing $\vec{l}_+ = A\vec{u} + \vec{b}$, with $\vec{b}$ orthogonal to $\vec{u}$, it follows $b_\mu b^\mu = A^2$ from the condition of $\vec{l}_+$ being null. On the other hand we have the decomposition $A\vec{u} + \vec{b} = \vec{l}_+ = \vec{n} + \vec{m}$. Multiplying by $\vec{u}$ we immediately get $A = \cosh\alpha - \cos\varphi\sinh\alpha$, and, since $\phi = \dot{a}(t)$ only depends on t,

$$\vec{l}_+(\phi) = (\cosh\alpha - \cos\varphi\sinh\alpha)\,\ddot{a}(t). \tag{3.4.9}$$

The following expression for $G^{(4)}_{\mu\nu} l^\mu_+ l^\nu_+$ follows directly from $\vec{l}_+ = A\vec{u} + \vec{b}$ and (3.4.5), (3.4.6),

$$\begin{aligned} G^{(4)}_{\mu\nu} l^\mu_+ l^\nu_+ &= A^2(\mu + p) \\ &= 2\,(\cosh\alpha - \cos\varphi\sinh\alpha)^2 \left(\frac{\dot{a}^2(t) + k}{a^2(t)} - \frac{\ddot{a}(t)}{a(t)}\right). \end{aligned} \tag{3.4.10}$$

Inserting (3.4.9) and (3.4.10) into (3.4.8) and dividing by $2A^2\cosh\alpha$ (which is positive) we find the equivalent condition

$$\left(\frac{1}{\cosh\alpha\,(\cosh\alpha - \cos\varphi\sinh\alpha)} - 1\right)\ddot{a}(t) + \frac{\dot{a}^2(t) + k}{a(t)} \geq 0, \tag{3.4.11}$$

and non-zero somewhere. The dependence on S only arises through the function $f(\alpha, \varphi) = \cosh\alpha(\cosh\alpha - \cos\varphi\sinh\alpha)$. Rewriting this as $f = 1/2(1 + \cosh(2\alpha) - \cos\varphi\sinh(2\alpha))$ it is immediate to show that f takes all values in $(1/2, +\infty)$. Hence, $[\cosh\alpha\,(\cosh\alpha - \cos\varphi\sinh\alpha)]^{-1} - 1$ takes all values between -1 and 1. In order to satisfy (3.4.11) on all this range, it is necessary and sufficient that the three inequalities in (3.4.7) are satisfied. ■

The following corollary gives a particularly interesting case where all the conditions of Theorem 3.4.6 are satisfied.

Corollary 3.4.7 *Consider a FLRW spacetime $(M, g^{(4)}_{FLRW})$ satisfying the NEC. If $\ddot{a}(t) > 0$, then there exists no stable MOTS in any spacelike hypersurface of $(M, g^{(4)}_{FLRW})$*

Proof. The null energy condition gives $0 \leq \mu + p = 2\left(\frac{\dot{a}^2(t)+k}{a^2(t)} - \frac{\ddot{a}(t)}{a(t)}\right)$. This implies the first and third inequalities in (3.4.7) if $\ddot{a} > 0$. The remaining condition $-\frac{\dot{a}^2(t)+k}{a(t)} \leq \ddot{a}$ is also obviously satisfied provided $\ddot{a} > 0$. ■

3.4.2 A consequence of the geometric construction of S_τ

We have emphasized at the beginning of this section that the restrictions obtained directly by the geometric procedure of moving S along $\vec{\xi}$ and then back to Σ are intuitively clear but typically weaker than those obtained by using elliptic theory results. There are some cases, however, where the reverse actually holds, and the geometric construction provides stronger results. We will present one of these cases in this subsection.

Corollary 3.4.3 gives restrictions on $\xi_\mu l_+^\mu|_S$ for Killing vectors and homotheties in spacetimes satisfying the NEC, provided $\vec{\xi}$ is future or past directed everywhere. However, when W vanishes identically, the result only gives useful information in the strictly stable case. The reason is that $W \equiv 0$ implies $L_{\vec{m}} Q \equiv 0$ and, for marginally stable MOTS (i.e. when the principal eigenvalue of $L_{\vec{m}}$ vanishes), the maximum principle is not strong enough to conclude that Q must have a sign. There is at least one case where marginally stable MOTS play an important role, namely after a jump in the outermost MOTS in a (3+1) foliation of the spacetime (see [1] for details). As we will see next, the geometric construction does give restrictions in this case even when W vanishes identically.

Theorem 3.4.8 *Consider a spacetime $(M, g^{(4)})$ possessing a Killing vector or a homothety $\vec{\xi}$ and satisfying the NEC. Suppose M contains a compact spacelike hypersurface $\tilde{\Sigma}$ with boundary consisting in the disjoint union of a weakly outer trapped surface $\partial^-\tilde{\Sigma}$ and an outer untrapped surface $\partial^+\tilde{\Sigma}$ (neither of which are necessarily connected) and take $\partial^+\tilde{\Sigma}$ as a barrier with interior $\tilde{\Sigma}$. Without loss of generality, assume that $\tilde{\Sigma}$ is defined locally by a level function $T = 0$ with $T > 0$ to the future of $\tilde{\Sigma}$ and let S be the outermost MOTS which is bounding with respect to $\partial^+\tilde{\Sigma}$. If $\vec{\xi}(T) \leq 0$ on some spacetime neighbourhood of S, then $\xi^\mu l_\mu^+ \leq 0$ everywhere on S.*

Remark 1. As usual, the theorem still holds if all the inequalities involving $\vec{\xi}$ are reversed. □

Remark 2. The simplest way to ensure that $\vec{\xi}(T) \leq 0$ on some neighbourhood of S is by imposing a condition merely on S, namely $\xi_\mu n^\mu|_S > 0$, because then $\vec{\xi}$ lies strictly below $\tilde{\Sigma}$ on S and this property is obviously preserved sufficiently near S (i.e. $\vec{\xi}$ points strictly below the level set of T on a sufficiently small spacetime neighbourhood of S). We prefer imposing directly the condition $\vec{\xi}(T) \leq 0$ on a spacetime neighbourhood of S because this allows $\vec{\xi}|_S$ to be

tangent to Σ. □

Proof. First note that the hypersurface $\tilde{\Sigma}$ satisfies the assumptions of Theorem 2.2.31 which ensures that an outermost MOTS S which is bounding with respect to $\partial^+\tilde{\Sigma}$ does exist and, therefore, no weakly outer trapped surface can penetrate in its exterior region. Then, the idea is precisely to use the geometric procedure described above to construct S_τ and use the fact that S is the outermost bounding MOTS to conclude that S_τ ($\tau > 0$) cannot have points outside S. Here we move S a small but finite amount τ, in contrast to the elliptic results before, which only involved infinitesimal displacements. We want to have information on the sign of the outer expansion of S_τ in order to make sure that a weakly outer trapped surface forms. The first part of the displacement is along $\vec{\xi}$ and gives S'_τ. Let us first see that all these surfaces are MOTS. For Killing vectors, this follows at once from symmetry arguments. For homotheties we have the identity

$$\delta_{\vec{\xi}}\theta^+ = \left(-\frac{1}{2}l_-^\alpha \mathscr{L}_{\vec{\xi}} l'_{+\alpha}(\tau) - 2C\right)\theta^+, \tag{3.4.12}$$

which follows directly from (3.3.1) with $\vec{\eta} = \vec{l}_+$ after using $l_+^\mu \mathscr{L}_{\vec{\xi}} l'_{+\mu}(\tau) = \frac{1}{2}a_{\mu\nu}(\vec{\xi})l_+^\mu l_+^\nu = 0$, see (3.3.8). Expression (3.4.12) holds for each one of the surfaces $\{S'_\tau\}$, independently of them being MOTS or not. Since this variation vanishes on MOTS and the starting surface S has this property, it follows that each surface S'_τ ($\tau > 0$) is also a MOTS. Moving back to $\tilde{\Sigma}$ along the null hypersurface introduces, via the Raychaudhuri equation (2.2.13), a non-positive term $N_S W$ in the outer null expansion, provided the motion is to the future. Hence, S_τ for small but finite $\tau > 0$ is a weakly outer trapped surface provided $\vec{\xi}$ moves to the past of $\tilde{\Sigma}$. This is ensured if $\vec{\xi}(T) \leq 0$ near S, because T cannot become positive for small enough τ. On the other hand, since a point $\mathfrak{p} \in S$ moves initially along the vector field $\nu = \vec{\xi} - N_S\vec{l}_+ = Q\vec{m} + \vec{Y}^{\|}$, where $Q = \xi_\mu l_+^\mu$ as usual, it follows that $Q > 0$ somewhere implies (for small enough τ) that the bounding weakly outer trapped surface S_τ has a portion lying strictly to the outside of S which, due to Theorem 2.2.31 by Andersson and Metzger, is a contradiction to S being the outermost bounding MOTS. Hence $Q \leq 0$ everywhere and the theorem is proven. ■

It should be remarked that the assumption of $\vec{\xi}$ being a Killing vector or a homothety is important for this result. Trying to generalize it for instance to conformal Killings fails in general because then the right hand side of equation

(3.4.12) has an additional term $2\vec{l}_+(\phi)$, not proportional to θ^+. This means that moving a MOTS along a conformal Killing does not lead to another MOTS in general. The method can however, still give useful information if $\vec{l}_+(\phi)$ has the appropriate sign, so that S'_τ is in fact weakly outer trapped. We omit the details.

An immediate consequence of the finite construction of S_τ is the extension of point (ii) of Corollary 3.4.4 to locally outermost MOTS.

Theorem 3.4.9 *Let $(M, g^{(4)})$ be a spacetime satisfying NEC and admitting a causal Killing vector or homothety $\vec{\xi}$ which is future (past) directed on a locally outermost MOTS $S \subset \Sigma$. Then $\vec{\xi} \propto \vec{l}_+$ everywhere on S.*

Proof. As before, let Σ be defined locally by a level function $T = 0$ with $T > 0$ to the future of Σ. Assume that $\vec{\xi}$ is past directed (the future directed case is similar). Then, the assumption $\vec{\xi}(T) \leq 0$ on some spacetime neighbourhood of S of Theorem 3.4.8 is automatically satisfied. Then we can use the finite construction therein to find a weakly outer trapped surface which, due to the fact that $\vec{\xi}$ is causal (and past directed), does not penetrate in the interior part of the two-sided neighbourhood of S. In fact, this new trapped surface will have points strictly outside S if on some point of S $\vec{\xi} \not\propto \vec{l}_+$ which proves the result. ■

Finally, Theorem 3.4.9 together with Theorem 2.2.31 lead to the following result.

Theorem 3.4.10 *Consider a spacelike hypersurface (Σ, g, K) possibly with boundary in a spacetime satisfying the NEC and possessing a Killing vector or a homothety $\vec{\xi}$ with squared norm $\xi_\mu \xi^\mu = -\lambda$. Assume that Σ possesses a barrier S_b with interior Ω_b which is outer untrapped with respect to the direction pointing outside of Ω_b.*

Consider any surface S which is bounding with respect to S_b. Let us denote by Ω the exterior of S in Ω_b. If S is weakly outer trapped and $\Omega \subset \{\lambda > 0\}$, then λ cannot be strictly positive on any point $\mathfrak{p} \in S$.

Remark. When *weakly outer trapped surface* is replaced by the stronger condition of being a *weakly trapped surface with non-vanishing mean curvature*, then this theorem can be proven by a simple argument based on the first variation of area [82]. In that case, the assumption of S being bounding becomes unnecessary. It would be interesting to know if Theorem 3.4.10 holds for arbitrary weakly outer trapped surfaces, not necessarily bounding. □

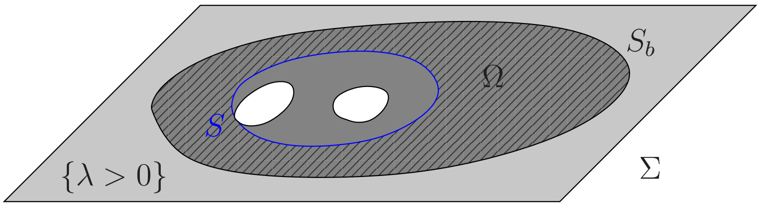


Figure 3.4: Theorem 3.4.10 excludes the possibility pictured in this figure, where S (in blue) is a weakly outer trapped surface which is bounding with respect to the outer trapped barrier S_b. The grey (both light and dark) regions represent the region where $\lambda > 0$. The dark grey region represents the interior of S_b, while the striped area corresponds to Ω, which is the exterior of S in Ω_b.

Proof. We argue by contradiction. Suppose a weakly outer trapped surface S satisfying the assumptions of the theorem and with $\lambda > 0$ at some point. Theorem 2.2.31 implies that an outermost MOTS $\partial^{top}T^+$ which is bounding with respect to S_b exists in the closure of the exterior Ω of S in Ω_b. In particular, $\partial^{top}T^+$ is a locally outermost MOTS. The hypothesis $\Omega \subset \{\lambda > 0\}$ implies that the vector $\vec{\xi}$ is causal everywhere on $\partial^{top}T^+$, either future or past directed. Moreover, the fact that $\lambda > 0$ on some point of S implies that the Killing vector is timelike in some non-empty set of $\partial^{top}T^+$, which contradicts Theorem 3.4.9. ■

The following result is a particularization of Theorem 3.4.10 to the case when the hypersurface Σ possesses an asymptotically flat end.

Theorem 3.4.11 *Let (Σ, g, K) be a spacelike hypersurface in a spacetime satisfying the NEC and possessing a Killing vector or homothety $\vec{\xi}$. Suppose that Σ possesses an asymptotically flat end Σ_0^∞.*

Consider any bounding surface S (see Definition 2.3.6). Let us denote by Ω the exterior of S in Σ. If S is weakly outer trapped and $\Omega \subset \{\lambda > 0\}$, then λ cannot be strictly positive on any point $\mathfrak{p} \in S$.

Proof. The result is a direct consequence of Theorem 3.4.10. ■

Two immediate corollaries follow.

Corollary 3.4.12 *Consider a spacelike hypersurface (Σ, g, K) in a spacetime satisfying the NEC and possessing a Killing vector or a homothety $\vec{\xi}$. Assume that Σ has a selected asymptotically flat end Σ_0^∞ and $\lambda > 0$ everywhere on Σ. Then there exists no bounding weakly outer trapped surface in Σ.*

Corollary 3.4.13 *Let (Σ, g, K) be a spacelike hypersurface of the Minkowski spacetime. Then there exists no bounding weakly outer trapped surface in Σ.*

The second Corollary is obviously a particular case of the first one because the vector ∂_t in Minkowskian coordinates is strictly stationary everywhere, in particular on Σ. The non-existence result of a bounding weakly outer trapped surface in a Cauchy surface of Minkowski spacetime is however, well-known as this spacetime is obviously regular predictable (see [65] for definition) and then the proof of Proposition 9.2.8 in [65] gives the result.

So far, all the results we have obtained require that the quantity $L_m Q$ does not change sign on the MOTS S. In the next section we will relax this condition.

3.5 Results regardless of the sign of $L_{\vec{m}}Q$

When $L_{\vec{m}}Q$ changes sign on S, the elliptic methods exploited in the previous section lose their power. Moreover, for sufficiently small τ, the surface S_τ defined by the geometric construction above necessarily fails to be weakly outer trapped. Thus, obtaining restrictions in this case becomes a much harder problem.

However, for locally outermost MOTS S, an interesting situation arises when S_τ lies partially outside S and happens to be weakly outer trapped in that exterior region. More precisely, if a connected component of the subset of S_τ which lies outside S turns out to have non-positive outer null expansion, then using a smoothing result by Kriele and Hayward [77], we will be able to construct a new weakly outer trapped surface outside S, thus leading to a contradiction with the fact that S is locally outermost (or else giving restrictions on the generator $\vec{\xi}$).

The result by Kriele and Hayward states, in rough terms, that given two surfaces which intersect on a curve, a new smooth surface can be constructed lying outside the previous ones in such a way that the outer null expansion does not increase in the process. The precise statement is as follows.

Lemma 3.5.1 (Kriele, Hayward, 1997 [77]) *Let $S_1, S_2 \subset \Sigma$ be smooth two-sided surfaces which intersect transversely on a smooth curve γ. Suppose that the exterior regions of S_1 and S_2 are properly defined in Σ and let U_1 and U_2 be respectively tubular neighbourhoods of S_1 and S_2 and U_1^- and U_2^- their interior parts. Assume it is possible to choose one connected component of each set $S_1 \setminus \gamma$ and $S_2 \setminus \gamma$, say S_1^+ and S_2^+ respectively, such that $S_1^+ \cap U_2^- = \emptyset$ and $S_2^+ \cap U_1^- = \emptyset$. Then, for any neighbourhood V of γ in Σ there exists a smooth surface $\tilde{S}$ and a continuous and piecewise smooth bijection $\Phi\colon S_1^+ \cup S_2^+ \cup \gamma \to \tilde{S}$ such that*

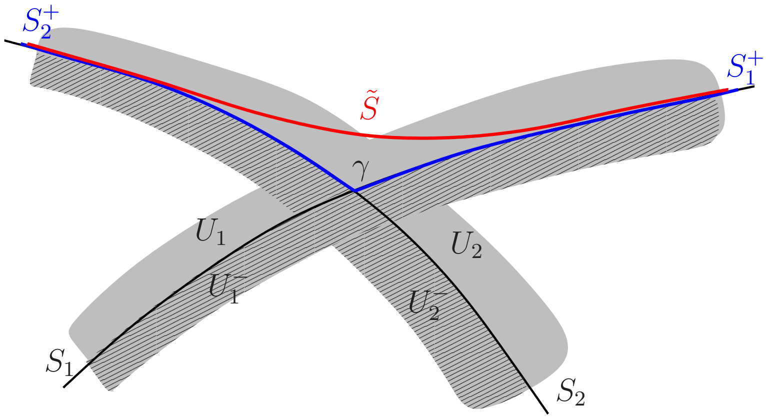


Figure 3.5: The figure represents the two surfaces S_1 and S_2 which intersects in a curve γ, (where one dimension has been suppressed). The two intersecting grey regions are the tubular neighbourhoods U_1 and U_2 and, inside them, the stripped regions represents their interior parts, U_1^- and U_2^-. The sets S_1^+ and S_2^+, in blue color, are then taken to be the connected components of $S_1 \setminus \gamma$ and $S_2 \setminus \gamma$ which do not intersect U_2^- and U_1^-, respectively. Finally, the red line represents the smooth surface $\tilde{S}$ which has smaller θ^+ than S_1 and S_2..

1. $\Phi(\mathfrak{p}) = \mathfrak{p}$, $\forall \mathfrak{p} \in \left(S_1^+ \cup S_2^+\right) \setminus V$
2. $\theta^+[\tilde{S}]\Big|_{\Phi(\mathfrak{p})} \leq \theta^+[S_A^+]\big|_{\mathfrak{p}}$ $\forall \mathfrak{p} \in S_A^+$ *($A = 1, 2$).*

Moreover $\tilde{S}$ lies in the connected component of $V \setminus \left(S_1^+ \cup S_2^+ \cup \gamma\right)$ lying in the exterior regions of both S_1 and S_2.

Remark. It is important to emphasize that the statement of this result is slightly different from the one appearing in the original paper [77] by Kriele and Hayward. Indeed, the assumptions made in [77] are rather ambiguous and restrictive in the sense that the outer normals of S_1 and S_2 are required to form an angle (defined only by a figure), not smaller than 90 degrees. This condition is not necessary for the lemma to work. This result also appears quoted in [4] where the assumptions are wrongly formulated (although the result is properly used throughout the paper). In our paper [26], where Lemma 3.5.1 is also formulated, the hypotheses are incomplete as well. □

This result will allow us to adapt the arguments above without having to assume that $L_{\vec{m}}Q$ has a constant sign on S. The argument will be again by contradiction, i.e. we will assume a locally outermost MOTS S and, under suitable

circumstances, we will be able to find a new weakly outer trapped surface lying outside S. Since the conditions are much weaker than in the previous section, the conclusion is also weaker. It is, however, fully general in the sense that it holds for any vector field $\vec{\xi}$ on S. Recall that Z is defined in equation (3.4.1).

Theorem 3.5.2 *Let S be a locally outermost MOTS in a spacelike hypersurface Σ of a spacetime $(M, g^{(4)})$. Denote by U_0 a connected component of the set $\{\mathfrak{p} \in S; \xi_\mu l_+^\mu|_\mathfrak{p} > 0\}$. Assume $U_0 \neq \emptyset$ and that its boundary $\gamma \equiv \partial^{top} U_0$ is either empty, or it satisfies that the function $\xi_\mu l_+^\mu$ has a non-zero gradient everywhere on γ, i.e. $d(\xi_\mu l_+^\mu)|_\gamma \neq 0$.*

Then, there exists a point $\mathfrak{p} \in \overline{U_0}$ such that $Z|_\mathfrak{p} \geq 0$.

Proof. As mentioned, we will use a contradiction argument. Let us therefore assume that

$$Z|_\mathfrak{p} < 0, \quad \forall \mathfrak{p} \in \overline{U_0}. \tag{3.5.1}$$

The aim is to construct a weakly outer trapped surface near S and outside of it. This will contradict the condition of S being locally outermost.

First of all we observe that Z cannot be negative everywhere on S, because then Theorem 3.4.2 (recall that outermost MOTS are always stable) would imply $Q \equiv (\xi_\mu l_+^\mu) < 0$ everywhere and U_0 would be empty against hypothesis. Consequently, under (3.5.1), U_0 cannot coincide with S and $\gamma \equiv \partial^{top} U_0 \neq \emptyset$. Since $Q|_\gamma = 0$ and, by assumption, $dQ|_\gamma \neq 0$ it follows that γ is a smooth embedded curve. Taking μ to be a local coordinate on γ, it is clear that $\{\mu, Q\}$ are coordinates of a neighbourhood of γ in S. We will coordinate a small enough neighbourhood of γ in Σ by Gaussian coordinates $\{u, \mu, Q\}$ such that $u = 0$ on S and $u > 0$ on its exterior.

By moving S along $\vec{\xi}$ a finite but small parametric amount τ and back to Σ with the outer null geodesics, as described in Section 3.2, we construct a family of surfaces $\{S_\tau\}_\tau$. The curve that each point $\mathfrak{p} \in S$ describes via this construction has tangent vector $\vec{\nu} = Q\vec{m} + \vec{Y}^\parallel|_S$ on S. In a small neighbourhood of γ, the normal component of this vector, i.e. $Q\vec{m}$, is smooth and only vanishes on γ. This implies that for small enough τ, S_τ are graphs over S near γ. We will always work on this neighbourhood, or suitable restrictions thereof. In the Gaussian coordinates above, this graph is of the form $\{u = \hat{u}(\mu, Q, \tau), \mu, Q\}$. Since the normal unit vector to S is simply $\vec{m} = \partial_u$ in these coordinates and the normal component of $\vec{\nu}$ is $Q\vec{m}$, the graph function $\hat{u}$ has the following Taylor expansion

$$\hat{u}(\mu, Q, \tau) = Q\tau + O(\tau^2). \tag{3.5.2}$$

Our next aim is to use this expansion to conclude that the intersection of S and S_τ near γ is an embedded curve γ_τ for all small enough τ. To do that we will apply the implicit function theorem for functions to the equation $\hat{u} = 0$. It is useful to introduce a new function $v(\mu, Q, \tau) = \frac{\hat{u}(\mu,Q,\tau)}{\tau}$, which is still smooth (thanks to (3.5.2)) and vanishes at $\tau = 0$ only on the curve γ. Moreover, its derivative with respect to Q is nowhere zero on γ, in fact $\frac{\partial v}{\partial Q}\Big|_{(\mu,0,0)} = 1$ for all μ. The implicit function theorem implies that there exist a unique function $Q = \varphi(\mu, \tau)$ which solves the equation $v(\mu, Q, \tau) = 0$, for small enough τ. Obviously, this function is also the unique solution near γ of $\hat{u}(\mu, Q, \tau) = 0$ for $\tau > 0$. Consequently, the intersection of S and S_τ ($\tau > 0$) lying in the neighbourhood of γ where we are working on is an embedded curve γ_τ. Since γ separates S into two or more connected components, the same is true for γ_τ for small enough τ (note that γ need not be connected and the number of connected components of $S \setminus \gamma$ may be bigger than two). Recall that γ is the boundary of a connected set U_0. Hence, by construction, there is only one connected component of $S_\tau \setminus \gamma_\tau$ which has $v(\mu, Q, \tau) > 0$ near γ (i.e. that lies in the exterior of S near γ). Let us denote it by S_τ^+. S_τ^+ in fact lies fully outside of S, not just in a neighborhood of γ, as we see next. First of all, note that $Q > 0$ on U_0. We have just seen that γ_τ is a continuous deformation of γ. Let us denote by U_τ the domain in S obtained by deforming U_0 when the boundary moves from γ to γ_τ (See Figure 3.6). It is obvious that S_τ^+ is obtained by moving U_τ first along $\vec{\xi}$ an amount τ and then back to Σ by null hypersurfaces. The closed subset of U_τ lying outside the tubular neighbourhood where we applied the implicit function theorem is, by construction a proper subset of U_0. Consequently, on this closed set Q is uniformly bounded below by a positive constant. Given that Q is the first order term of the normal variation, all these points move outside of S. This proves that S_τ^+ is fully outside S for sufficiently small τ. Incidentally this also shows that S_τ^+ is a graph over U_τ.

The next aim is to show that the outer null expansion of S_τ is non-positive everywhere on S_τ^+. To that aim, we will prove that, for small enough τ, Z is strictly negative everywhere on U_τ. Since Z is the first order term in the variation of θ^+, this implies that the outer null expansion of S_τ^+ satisfies $\theta^+[S_\tau^+] < 0$ for $\tau > 0$ small enough.

By assumption (3.5.1), Z is strictly negative on U_0. Therefore, this quantity is automatically negative in the portion of U_τ lying in U_0 (in particular, outside the tubular neighbourhood where we applied the implicit function theorem). The only difficulty comes from the fact that γ_τ may move outside U_0 at some points and we only have information on the sign of Z on $\overline{U_0}$. To address this issue, we

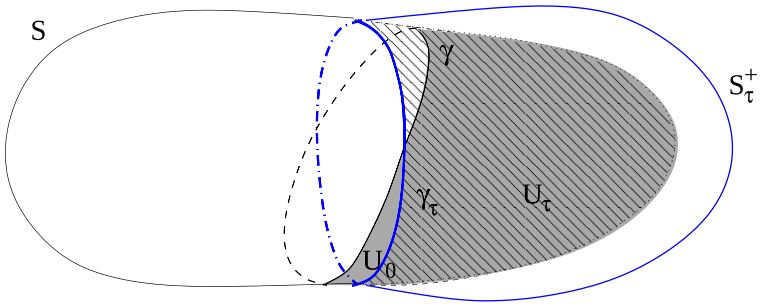


Figure 3.6: The figure represents both intersecting surfaces S and S_τ^+ together with the curves γ and γ_τ. The shaded region corresponds to U_0 and the stripped region to U_τ.

first notice that Q defines a distance function to γ (because Q vanishes on γ and its gradient is nowhere zero). Consequently, the fact that Z is strictly negative on γ (by assumption (3.5.1)) and that this curve is compact imply that there exists a $\delta > 0$ such that, inside the tubular neighbourhood of γ, $|Q| < \delta$ implies $Z < 0$. Moreover, the function $Q = \varphi(\mu, \tau)$, which defines γ_τ, is such that it vanishes at $\tau = 0$ and depends smoothly on τ. Since μ takes values on a compact set, it follows that for each $\delta' > 0$, there exists an $\epsilon(\delta') > 0$, independent of μ such that $|\tau| < \epsilon(\delta')$ implies $|Q| = |\varphi(\mu, \tau)| < \delta'$. By taking $\delta' = \delta$, it follows that, for $|\tau| < \epsilon(\delta)$, U_τ is contained in a δ-neighbourhood of U_0 (with respect to the distance function Q) and consequently $Z < 0$ on this set, as claimed. We restrict to $0 < \tau < \epsilon(\delta)$ from now on.

Summarizing, so far we have shown that S_τ^+ lies fully outside S and has $\theta^+[S_\tau^+] < 0$. The final task is to use Lemma 3.5.1 to construct a weakly outer trapped surface strictly outside S. Denote by S_τ^* the complement of U_τ in S, which may have several connected components. For any connected component γ_τ^i of γ_τ there exists a neighbourhood $W_{\tau,i}^*$ of γ_τ^i in $S_\tau^* \subset S$ which lies in the exterior of S_τ (because the intersection between S and S_τ is transverse). Similarly, there is a connected neighbourhood $W_{\tau,i}^+$ of γ_τ^i in $S_\tau^+ \subset S_\tau$ which lies in the exterior of S. The smoothing argument of Lemma 3.5.1 can be therefore applied locally on each union $W_{\tau,i}^* \cup \gamma_\tau^i \cup W_{\tau,i}^+$ to produce a weakly outer trapped surface $\tilde{S}$ which lies outside S, leading a contradiction. This surface $\tilde{S}$ is constructed in such a way that $\tilde{S} = S_\tau^*$ in $S_\tau^* \setminus \left(\underset{i}{\cup} W_{\tau,i}^* \right)$ and $\tilde{S} = S_\tau^+$ in $S_\tau^+ \setminus \left(\underset{i}{\cup} W_{\tau,i}^+ \right)$. ■

Remark. As usual, this theorem also holds if all the inequalities are reversed. Note that in this case U_0 is defined to be a connected component of the set

$\{\mathfrak{p} \in S; (\xi_\mu l_+^\mu)|_\mathfrak{p} < 0\}$. For the proof simply take $\tau < 0$ instead of $\tau > 0$ (or equivalently move along $-\vec{\xi}$ instead of $\vec{\xi}$). $\square$

Similarly as in the previous section, this theorem can be particularized to the case of conformal Killing vectors, as follows (recall that $Z = 2\vec{l}_+(\phi) + NW$ in the conformal Killing case, see Corollary 3.4.3).

Corollary 3.5.3 *Under the assumptions of Theorem 3.5.2, suppose that $\vec{\xi}$ is a conformal Killing vector with conformal factor ϕ (including homotheties $\phi = C$ and isometries $\phi = 0$).*

Then, there exists $\mathfrak{p} \in \overline{U_0}$ such that $2\vec{l}_+(\phi) + N_S(\Pi^+_{AB}\Pi^{+AB} + G^{(4)}_{\mu\nu} l_+^\mu l_+^\nu)|_\mathfrak{p} \geq 0$.

If the conformal Killing is in fact a homothety or a Killing vector and it is causal everywhere, the result can be strengthened considerably. The next result extends Corollary 3.4.4 in a suitable sense to the cases when the generator is not assumed to be either future or past everywhere. Since its proof requires an extra ingredient we write it down as a theorem.

Theorem 3.5.4 *In a spacetime $(M, g^{(4)})$ satisfying the NEC and admitting a Killing vector or homothety $\vec{\xi}$, consider a locally outermost MOTS S in a spacelike hypersurface Σ. Assume that $\vec{\xi}$ is causal on S and that $W = \Pi^+_{AB}\Pi^{+AB} + G^{(4)}_{\mu\nu} l_+^\mu l_+^\nu$ is non-zero everywhere on S. Define $U \equiv \{\mathfrak{p} \in S; (\xi_\mu l_+^\mu)|_\mathfrak{p} > 0\}$ and assume that this set is neither empty nor covers all of S. Then, on each connected component U_α of U there exist a point $\mathfrak{p} \in \partial^{top} U_\alpha$ with $d(\xi_\mu l_+^\mu)|_\mathfrak{p} = 0$.*

Remark 1. The same conclusion holds on the boundary of each connected components of the set $\{\mathfrak{p} \in S; (\xi_\mu l_+^\mu)|_\mathfrak{p} < 0\}$. This is obvious since $\vec{\xi}$ can be changed to $-\vec{\xi}$. $\square$

Remark 2. The case $\partial^{top} U = \emptyset$, excluded by assumption in this theorem, can only occur if $\vec{\xi}$ is future or past everywhere on S. Hence, this case is already included in Corollary 3.4.4. $\square$

Proof. We first show that on any point in U we have $N_S < 0$, which has as an immediate consequence that $N_S \leq 0$ on any point in $\overline{U}$. The former statement is a consequence of the decomposition $\vec{\xi} = N\vec{l}_+ + Q\vec{m} + \vec{Y}^\parallel$, where $Q = (\xi_\mu l_+^\mu)$. The condition that $\vec{\xi}$ is causal then implies $\xi_\mu \xi^\mu = 2N_S Q + Q^2 + {Y^\parallel}^2 \leq 0$. This can only happen at a point where $Q > 0$ (i.e. on U) provided $N_S < 0$ there. Moreover,

if at any point $\mathfrak{q}$ on the boundary $\partial^{top}U$ we have $N_S|_{\mathfrak{q}} = 0$, then necessarily the full vector $\vec{\xi}$ vanishes at this point. This implies, in particular, that the geometric construction of S_τ has the property that $\mathfrak{q}$ remains invariant.

Having noticed these facts, we will now argue by contradiction, i.e. we will assume that there exists a connected component U_0 of U such that $d(\xi_\mu l_+^\mu)|_{\partial^{top}U_0} \neq 0$ everywhere. In these circumstances, we can follow the same steps as in the proof of Theorem 3.5.2 to show that, for small enough τ the surface S_τ has a portion S_τ^+ lying in the exterior of S and which, in the Gaussian coordinates above, is a graph over a subset U_τ which is a continuous deformation of U_0. Moreover, the boundary of U_τ is a smooth embedded curve γ_τ. The only difficulty with this construction is that we cannot use $N_S W = Z < 0$ everywhere on $\overline{U}_0$, in order to conclude that $\theta^+[S_\tau^+] < 0$, as we did before. The reason is that there may be points on $\partial^{top}U_0$ where $N_S = 0$. However, as already noted, these points have the property that *do not move at all* by the construction of S_τ, i.e. the boundary γ_τ (which is the intersection of S and S_τ^+) can only move outside of U_0 at points where N_S is strictly negative. Hence on the interior points of U_τ we have $N_S < 0$ everywhere, for sufficiently small τ. Consequently the first order terms in the variation of θ^+, namely $Z = N_s W$, is strictly negative on all the interior points of U_τ. This implies that S_τ^+ has negative outer null expansion everywhere except possibly on its boundary γ_τ. By continuity, we conclude $\theta^+[S_\tau^+] \leq 0$ everywhere. The smoothing argument of the proof of Theorem 3.5.2 implies that a smooth weakly outer trapped surface can be constructed outside the locally outermost MOTS S. This gives a contradiction. Therefore, there exists $\mathfrak{p} \in \partial^{top}U_0$ such that $d(\xi_\mu l_+^\mu)|_{\mathfrak{p}} = 0$, as claimed. ■

Remark The assumption $dQ|_\gamma \neq 0$ is a technical requirement for using the smoothing argument of Lemma 3.5.1. This is why we had to include an assumption on $dQ|_\gamma$ in Theorem 3.5.2 and also that the conclusion of Theorem 3.5.4 is stated in terms of the existence of critical points for Q. If Lemma 3.5.1 could be strengthened so as to remove this requirement, then Theorem 3.5.4 could be rephrased as stating that any outermost MOTS in a region where there is a causal Killing vector (irrespective of its future or past character) must have at least one point where the shear and $G^{(4)}_{\mu\nu} l_+^\mu l_+^\nu$ vanish simultaneously.

In any case, the existence of critical points for a function in the boundary of *every* connected component of $\{Q > 0\}$ and *every* connected component of $\{Q < 0\}$ is obviously a highly non-generic situation. So, locally outermost MOTS in regions where there is a causal Killing vector or homothety can at most occur

under very exceptional circumstances. $\square$

Chapter 4

Weakly outer trapped surfaces in static spacetimes

4.1 Introduction

In the next two chapters we will concentrate on *static* spacetimes. As we have remarked in Chapter 1, one of the main aims of this thesis is to extend the uniqueness theorems for static black holes to static spacetimes containing MOTS. This chapter is devoted to obtaining a proper understanding of MOTS in static spacetimes, which will be essential to prove the uniqueness result in the next chapter.

The first answer to the question of whether the uniqueness theorems for static black holes extend to static spacetimes containing MOTS was given by Miao in 2005 [88], who proved uniqueness for the particular case of time-symmetric, asymptotically flat and vacuum spacelike hypersurfaces possessing a minimal compact boundary (note that in a time-symmetric slice compact minimal surfaces are MOTS and vice versa). This result generalized the classic uniqueness result of Bunting and Masood-ul-Alam [23] for vacuum static black holes which states the following.

Theorem 4.1.1 *Consider a vacuum spacetime $(M, g^{(4)})$ with a static Killing vector $\vec{\xi}$. Assume that $(M, g^{(4)})$ possesses a connected, asymptotically flat spacelike hypersurface (Σ, g, K) which is time-symmetric (i.e. $K = 0$, $\vec{\xi} \perp \Sigma$), has non-empty compact boundary $\partial\Sigma$ and is such that the static Killing vector $\vec{\xi}$ is causal on Σ and null only on $\partial\Sigma$.*
Then (Σ, g) is isometric to $\left(\mathbb{R}^3 \setminus B_{M_{Kr}/2}(0), (g_{Kr})_{ij} = \left(1 + \frac{M_{Kr}}{2|x|}\right)^4 \delta_{ij}\right)$ for some $M_{Kr} > 0$, i.e. the $\{t = 0\}$ slice of the Kruskal spacetime with mass M_{Kr} outside and including the horizon. Moreover, there exists a neighbourhood of Σ in M which is isometrically diffeomorphic to the closure of the domain of outer

communications of the Kruskal spacetime.

In other words, this theorem asserts that a time-symmetric slice Σ of a non-degenerate static vacuum black hole must be a time-symmetric slice of the Kruskal spacetime. Miao was able to reach the same conclusion under much weaker assumptions, namely by simply assuming that the boundary of Σ is a closed minimal surface. As in Bunting and Masood-ul-Alam's theorem, Miao's result deals with time-symmetric and asymptotically flat spacelike hypersurfaces embedded in static vacuum spacetimes. More precisely,

Theorem 4.1.2 *Consider a vacuum spacetime $(M, g^{(4)})$ with a static Killing vector $\vec{\xi}$. Assume that $(M, g^{(4)})$ possesses a connected, asymptotically flat spacelike hypersurface (Σ, g, K) which is time-symmetric and such that $\partial\Sigma$ is a (non-empty) compact minimal surface.*
Then (Σ, g) is isometric to $\left(\mathbb{R}^3 \setminus B_{M_{Kr}/2}(0), (g_{Kr})_{ij} = \left(1 + \frac{M_{Kr}}{2|x|}\right)^4 \delta_{ij}\right)$ for some $M_{Kr} > 0$, i.e. the $\{t = 0\}$ slice of the Kruskal spacetime with mass M_{Kr} outside and including the horizon. Moreover, there exists a neighbourhood of Σ in M which is isometrically diffeomorphic to the closure of the domain of outer communications of the Kruskal spacetime.

A key ingredient in Miao's proof was to show that the existence of a closed minimal surface implies the existence of an asymptotically flat end Σ^∞ with smooth topological boundary $\partial^{top}\Sigma^\infty$ such that $\vec{\xi}$ is timelike on Σ^∞ and vanishes on $\partial^{top}\Sigma^\infty$. Miao then proved that $\partial^{top}\Sigma^\infty$ coincides in fact with the minimal boundary $\partial\Sigma$ of the original manifold. Hence, the strategy was to reduce Theorem 4.1.2 to the Bunting and Massod-ul-Alam uniqueness theorem of black holes.

As a consequence of the static vacuum field equations the set of points where the Killing vector vanishes in a time-symmetric slice is known to be a totally geodesic surface. Totally geodesic surfaces are of course minimal and in this sense Theorem 4.1.2 is a generalization of Theorem 4.1.1. In fact, Theorem 4.1.1 allows us to rephrase Miao's theorem as follows: *No minimal surface can penetrate in the exterior region where the Killing vector is timelike in any time-symmetric and asymptotically flat slice of a static vacuum spacetime.* In this sense, Miao's result can be regarded as a confinement result for MOTS in time-symmetric slices of static vacuum spacetimes. Here, it is important to remark that a general confinement result of this type was already known when suitable global hypotheses in time are assumed in the spacetime. In this case, weakly outer trapped surfaces

must lie inside the black hole region (see e.g. Proposition 12.2.4 in [112]). Consequently, Theorem 4.1.2 can also be viewed as an extension of this result to the initial data setting (which drops completely all global assumptions in time) for the particular case of time-symmetric, static vacuum slices.

We aim to generalize Miao's theorem in three different directions. Firstly, we want to allow for non-vanishing matter as long as the NEC is satisfied. Secondly, the slices will no longer be required to be time-symmetric. In this situation the natural replacement for minimal surfaces are MOTS. And finally, we intend to relax the condition of asymptotic flatness to just assuming the presence of an outer untrapped surface (of course, this will not be possible for the uniqueness theorem, but it is possible when viewing Miao's result as a confinement result). The proof given by Miao relies strongly on the vacuum field equations, so we must resort to different methods. Obviously, a fundamental step for our purposes is a proper understanding of MOTS in static spacetimes.

In this chapter we explore the properties of MOTS in static spacetimes. The main result of this chapter is Theorem 4.4.1 which extends Theorem 4.1.2 as a confinement result for MOTS by asserting that no MOTS which are bounding can *penetrate* into the exterior region where the static Killing is timelike provided some hypotheses hold. In fact, this result for MOTS also holds for weakly outer trapped surfaces. It is important to note that Theorem 3.4.10 in the previous chapter already forbids the existence of weakly outer trapped surfaces whose exterior lies in the region where the Killing vector is timelike, and which penetrates into the timelike region (recall that the exterior of S does not contain S, by definition). However, this result does not exclude the existence of a weakly outer trapped surface penetrating into the timelike region but not lying entirely in the causal region. This is the situation we exclude in Theorem 4.4.1. The essential ingredients to prove this result will be a combination of the ideas that allowed us to prove Theorem 3.4.10 together with a detailed study of the properties of the boundary of the region where the static Killing is timelike. Besides a confinement result, Miao's theorem is also (and fundamentally) a uniqueness theorem. The generalization of Miao's result as a uniqueness result will be studied in the next chapter, where several of the results of the present chapter will be applied.

As we remarked in the introductory chapter, a general tendency in investigations involving stationary and static spacetimes over the years has been to relax the global hypotheses in time and work at the initial data level as much as possible. Good examples of this fact are the statements of Theorems 4.1.1 and 4.1.2 above, where the existence of a spacelike hypersurface with suitable properties is,

in fact, sufficient for the proof. Following this trend, all the results of this chapter will be proved by working directly on spacelike hypersurfaces, with no need of invoking a spacetime containing them. These spacelike hypersurfaces, considered as abstract objects on their own, will be called *initial data sets*. Some of these results generalize known properties of static spacetimes to the initial data setting and, consequently, can be of independent interest.

We finish this introduction with a brief summary of the chapter. In Section 4.2 we define initial data set as well as Killing initial data (KID). Then we introduce the so-called Killing form and give some of its properties. In Section 4.3 we discuss the implications of imposing staticity on a Killing initial data set and state a number of useful properties of the boundary of the set where the static Killing vector is timelike, which will be fundamental to prove Theorem 4.4.1. Some of the technical work required in this section is related to the fact that we are not a priori assuming the existence of a spacetime. Finally, Section 4.4 is devoted to stating and proving Theorem 4.4.1.

The results presented in this chapter have been published in [24], [25].

4.2 Preliminaries

4.2.1 Killing Initial Data (KID)

We start with the standard definition of initial data set [13].

Definition 4.2.1 *An* ***initial data set*** $(\Sigma, g, K; \rho, \mathbf{J})$ *is a 3-dimensional connected manifold* Σ*, possibly with boundary, endowed with a Riemannian metric* g*, a symmetric, rank-two tensor* K*, a scalar* ρ *and a one-form* $\mathbf{J}$ *satisfying the so-called constraint equations,*

$$\begin{aligned} 2\rho &= R^{\Sigma} + (tr_{\Sigma} K)^2 - K_{ij}K^{ij}, \\ -J_i &= \nabla^{\Sigma}{}_j (K_i{}^j - tr_{\Sigma} K \delta_i^j), \end{aligned}$$

where R^{Σ} *and* ∇^{Σ} *are respectively the scalar curvature and the covariant derivative of* (Σ, g) *and* $tr_{\Sigma} K = g^{ij} K_{ij}$*.*

For simplicity, we will often write (Σ, g, K) instead of $(\Sigma, g, K; \rho, \mathbf{J})$ when no confusion arises.

In the framework of the Cauchy problem for the Einstein field equations, Σ is a spacelike hypersurface of a spacetime $(M, g^{(4)})$, g is the induced metric and K is the second fundamental form. The **initial data energy density** ρ and **energy**

flux J are defined by $\rho \equiv G^{(4)}_{\mu\nu} n^\mu n^\nu$, $J_i \equiv -G^{(4)}_{\mu\nu} n^\mu e_i^\nu$, where $G^{(4)}_{\mu\nu}$ is the Einstein tensor of $g^{(4)}$, $\vec{n}$ is the unit future directed vector normal to Σ and $\{\vec{e}_i\}$ is a local basis for $\mathfrak{X}(\Sigma)$. When $\rho = 0$ and $\mathbf{J} = 0$, the initial data set is said to be **vacuum**.

As remarked in the previous section, we will regard initial data sets as abstract objects on their own, independently of the existence of a spacetime where they may be embedded, unless explicitly stated.

Consider for a moment a spacetime $(M, g^{(4)})$ possessing a Killing vector field $\vec{\xi}$ and let (Σ, g, K) be an initial data set in this spacetime. We can decompose $\vec{\xi}$ along Σ into a normal and a tangential component as

$$\vec{\xi} = N\vec{n} + Y^i \vec{e}_i \tag{4.2.1}$$

(see Figure 3.2), where $N = -\xi^\mu n_\mu$. Note that with this decomposition

$$\lambda \equiv -\xi_\mu \xi^\mu = N^2 - Y^2.$$

Inserting (4.2.1) into the Killing equations and performing a 3+1 splitting on (Σ, g, K) it follows (see [46], [13]),

$$\begin{aligned} 2NK_{ij} + 2\nabla^\Sigma_{(i} Y_{j)} &= 0, && (4.2.2)\\ \mathcal{L}_{\vec{Y}} K_{ij} + \nabla^\Sigma_i \nabla^\Sigma_j N &= N\left({R^\Sigma}_{ij} + \mathrm{tr}_\Sigma K K_{ij} - 2K_{il}K^l_j - \tau_{ij} \right. \\ &\qquad \left. + \frac{1}{2} g_{ij} (\mathrm{tr}_\Sigma \tau - \rho) \right), && (4.2.3) \end{aligned}$$

where the parentheses in (4.2.2) denotes symmetrization, $\tau_{ij} \equiv G^{(4)}_{\mu\nu} e_i^\mu e_j^\nu$ are the remaining components of the Einstein tensor and $\mathrm{tr}_\Sigma \tau = g^{ij}\tau_{ij}$. Thus, the following definition of Killing initial data becomes natural [13].

Definition 4.2.2 *An initial data set $(\Sigma, g, K; \rho, \mathbf{J})$ endowed with a scalar N, a vector $\vec{Y}$ and a symmetric tensor τ_{ij} satisfying equations (4.2.2) and (4.2.3) is called a* ***Killing initial data*** *(KID).*

In particular, if a KID has $\rho = 0$, $\mathbf{J} = 0$ and $\tau = 0$ then it is said to be a **vacuum KID**.

A point $\mathfrak{p} \in \Sigma$ where $N = 0$ and $\vec{Y} = 0$ is a **fixed point**. This name is motivated by the fact that when the KID is embedded into a spacetime with a local isometry, the corresponding Killing vector $\vec{\xi}$ vanishes at $\mathfrak{p}$ and the isometry has a fixed point there.

A natural question regarding KID is whether they can be embedded into a spacetime $(M, g^{(4)})$ such that N and $\vec{Y}$ correspond to a Killing vector $\vec{\xi}$. The

simplest case where existence is guaranteed involves "transversal" KID, i.e. when $N \neq 0$ everywhere. Then, the following spacetime, called **Killing development** of (Σ, g, K), can be constructed

$$\left(\Sigma \times \mathbb{R}, \quad g^{(4)} = -\hat{\lambda} dt^2 + 2\hat{Y}_i dt dx^i + \hat{g}_{ij} dx^i dx^j\right) \tag{4.2.4}$$

where

$$\hat{\lambda}(t, x^i) \equiv (N^2 - Y^i Y_i)(x^i), \quad \hat{g}_{ij}(t, x^k) \equiv g_{ij}(x^k), \quad \hat{Y}^i(t, x^j) \equiv Y^i(x^j). \tag{4.2.5}$$

Notice that ∂_t is a complete Killing field with orbits diffeomorphic to $\mathbb{R}$ which, when evaluated on $\Sigma \equiv \{t = 0\}$ decomposes as $\partial_t = N\vec{n} + Y^i \vec{e}_i$, in agreement with (4.2.1). The Killing development is the unique spacetime with these properties. Further details can be found in [13]. Notice also that the Killing development can be constructed for any connected subset of Σ where $N \neq 0$ everywhere.

We will finish this subsection by giving the definition of asymptotically flat KID, which is just the same as for asymptotically flat spacelike hypersurface *but adding* the suitable decays for the quantities N and $\vec{Y}$.

Definition 4.2.3 *A KID* $(\Sigma, g, K; N, \vec{Y}, \tau)$ *is* **asymptotically flat** *if* $\Sigma = \mathcal{K} \cup \Sigma^\infty$*, where* $\mathcal{K}$ *is a compact set and* $\Sigma^\infty = \bigcup_a \Sigma_a^\infty$ *is a finite union with each* Σ_a^∞*, called an asymptotic end, being diffeomorphic to* $\mathbb{R}^3 \setminus \overline{B_{R_a}}$*, where* B_{R_a} *is an open ball of radius* R_a*. Moreover, in the Cartesian coordinates* $\{x^i\}$ *induced by the diffeomorphism, the following decay holds*

$$\begin{array}{ll} N - A_a = O^{(2)}(1/r), & g_{ij} - \delta_{ij} = O^{(2)}(1/r), \\ Y^i - C_a^i = O^{(2)}(1/r), & K_{ij} = O^{(2)}(1/r^2). \end{array}$$

where A_a *and* $\{C_a^i\}_{i=1,2,3}$ *are constants such that* $A_a^2 - \delta_{ij} C_a^i C_a^j > 0$ *for each* a*, and* $r = (x^i x^j \delta_{ij})^{1/2}$*.*

Remark. The condition on the constants A_a, C_a^i is imposed to ensure that the KID is timelike near infinity on each asymptotic end. □

4.2.2 Killing Form on a KID

A useful object in spacetimes with a Killing vector $\vec{\xi}$ is the two-form $\nabla_\mu \xi_\nu$, usually called **Killing form** or also Papapetrou field. This tensor will play a relevant role below. Since we intend to work directly on the initial data set, we need to define a suitable tensor on (Σ, g, K) which corresponds to the Killing form whenever

a spacetime is present. Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a KID in $(M, g^{(4)})$. Clearly we need to restrict and decompose $\nabla_\mu \xi_\nu$ onto $(\Sigma, g, K; N, \vec{Y}, \tau)$ and try to get an expression in terms of N and $\vec{Y}$ and its spatial derivatives. In order to use (4.2.1) we first extend $\vec{n}$ to a neighbourhood of Σ as a timelike unit and hypersurface orthogonal, but otherwise arbitrary, vector field (the final expression we obtain will be independent of this extension), and define N and $\vec{Y}$ so that $\vec{Y}$ is orthogonal to $\vec{n}$ and (4.2.1) holds. Taking covariant derivatives we find

$$\nabla_\mu \xi_\nu = \nabla_\mu N n_\nu + N \nabla_\mu n_\nu + \nabla_\mu Y_\nu. \tag{4.2.6}$$

Notice that, by construction, $\nabla_\mu n_\nu|_\Sigma = K_{\mu\nu} - n_\mu a_\nu|_\Sigma$ where $a_\nu = n^\alpha \nabla_\alpha n_\nu$ is the acceleration of $\vec{n}$. To elaborate $\nabla_\mu Y_\nu$ we recall that ∇^Σ-covariant derivatives correspond to spacetime covariant derivatives projected onto Σ. Thus, from $\nabla^\Sigma_\mu Y_\nu \equiv h^\alpha_\mu h^\beta_\nu \nabla_\alpha Y_\beta$, where $h^\mu_\nu = \delta^\mu_\nu + n^\mu n_\nu$ is the projector orthogonal to $\vec{n}$, and expanding we find

$$\begin{aligned}
\nabla_\mu Y_\nu|_\Sigma &= \nabla^\Sigma_\mu Y_\nu - n_\mu \left(n^\alpha \nabla_\alpha Y_\beta\right) h^\beta_\nu - n_\nu \left(n^\beta \nabla_\alpha Y_\beta\right) h^\alpha_\mu + n_\mu n_\nu n^\alpha n^\beta \nabla_\alpha Y_\beta|_\Sigma \\
&= \nabla^\Sigma_\mu Y_\nu - n_\mu \left(n^\alpha \nabla_\alpha Y_\beta\right) h^\beta_\nu + n_\nu \left(Y^\beta \nabla_\alpha n_\beta\right) h^\alpha_\mu + n_\mu n_\nu n^\alpha n^\beta \nabla_\alpha Y_\beta|_\Sigma \\
&= \nabla^\Sigma_\mu Y_\nu - n_\mu \left(n^\alpha \nabla_\alpha Y_\beta\right) h^\beta_\nu + K_{\mu\alpha} Y^\alpha n_\nu + n_\mu n_\nu n^\alpha n^\beta \nabla_\alpha Y_\beta|_\Sigma,
\end{aligned}$$

Substitution into (4.2.6), using $\nabla_\mu N = \nabla^\Sigma_\mu N - n_\mu n^\alpha \nabla_\alpha N$, gives

$$\begin{aligned}
\nabla_\mu \xi_\nu\big|_\Sigma &= n_\nu \left(\nabla^\Sigma_\mu N + K_{\mu\alpha} Y^\alpha\right) - n_\mu \left(N a_\nu + n^\alpha h^\beta_\nu \nabla_\alpha Y_\beta\right) \\
&\quad + (\nabla^\Sigma_\mu Y_\nu + N K_{\mu\nu}) + n_\mu n_\nu \left(n^\alpha n^\beta \nabla_\alpha Y_\beta - n^\alpha \nabla_\alpha N\right)|_\Sigma.
\end{aligned} \tag{4.2.7}$$

The Killing equations then require $n^\alpha n^\beta \nabla_\alpha Y_\beta|_\Sigma = n^\alpha \nabla_\alpha N|_\Sigma$ and $\nabla^\Sigma_\mu N + K_{\mu\alpha} Y^\alpha|_\Sigma = N a_\mu + n^\alpha h^\beta_\mu \nabla_\alpha Y_\beta|_\Sigma$, so that (4.2.7) becomes, after using (4.2.2),

$$\nabla_\mu \xi_\nu|_\Sigma = n_\nu \left(\nabla^\Sigma_\mu N + K_{\mu\alpha} Y^\alpha\right) - n_\mu \left(\nabla^\Sigma_\nu N + K_{\nu\alpha} Y^\alpha\right) + \frac{1}{2}\left(\nabla^\Sigma_\mu Y_\nu - \nabla^\Sigma_\nu Y_\mu\right)\Big|_\Sigma. \tag{4.2.8}$$

This expression involves solely objects defined on Σ. However, it still involves four-dimensional objects. In order to work directly on the KID, we introduce an auxiliary four-dimensional vector space on each point of Σ as follows (we stress that we are *not* constructing a spacetime, only a Lorentzian vector space attached to each point on the KID).

At every point $\mathfrak{p} \in \Sigma$ define the vector space $V_\mathfrak{p} = T_\mathfrak{p}\Sigma \oplus \mathbb{R}$, and endow this space with the Lorentzian metric $g_0|_\mathfrak{p} = g|_\mathfrak{p} \oplus (-\delta)$, where δ is the canonical metric on $\mathbb{R}$. Let $\vec{n}$ be the unit vector tangent to the fiber $\mathbb{R}$. Having a metric we can

lower and raise indices of tensors in $T_{\mathfrak{p}}\Sigma \oplus \mathbb{R}$. In particular define $\mathbf{n} = g_0(\vec{n}, \cdot)$. Covariant tensors Q on $T_{\mathfrak{p}}\Sigma$ can be canonically extended to tensors of the same type on $V_{\mathfrak{p}} = T_{\mathfrak{p}}\Sigma \oplus \mathbb{R}$ (still denoted with the same symbol) simply by noticing that any vector in $V_{\mathfrak{p}}$ is of the form $\vec{X} + a\vec{n}$, where $\vec{X} \in T_{\mathfrak{p}}\Sigma$ and $a \in \mathbb{R}$. The extension is defined (for a type m covariant tensor) by $Q(\vec{X}_1 + a_1\vec{n}, \cdots, \vec{X}_m + a_m\vec{n}) \equiv Q(\vec{X}_1, \cdots, \vec{X}_m)$. In index notation, this extension will be expressed simply by changing Latin to Greek indices. It is clear that the collection of $(T_{\mathfrak{p}}\Sigma \oplus \mathbb{R}, g_0)$ at every $\mathfrak{p} \in \Sigma$ contains no more information than just (Σ, g). In particular, this construction allows us to redefine the energy conditions appearing in Chapter 3.2 at the initial data level. Let us give the definition of NEC for an initial data set.

Definition 4.2.4 *An initial data set (Σ, g, K) satisfies the* **null energy condition** *(NEC) if for all $\mathfrak{p} \in \Sigma$ the tensor $G^{(4)}_{\mu\nu} \equiv \rho n_\mu n_\nu + J_\mu n_\nu + n_\mu J_\nu + \tau_{\mu\nu}$ on $T_{\mathfrak{p}}\Sigma \times \mathbb{R}$ satisfies that $G^{(4)}_{\mu\nu} k^\mu k^\nu|_{\mathfrak{p}} \geq 0$ for any null vector $\vec{k} \in T_{\mathfrak{p}}\Sigma \oplus \mathbb{R}$.*

Motivated by (4.2.8), we can define the Killing form directly in terms of objects on the KID

Definition 4.2.5 *The* **Killing form on a KID** *is the 2-form $F_{\mu\nu}$ defined on $(T_{\mathfrak{p}}\Sigma \oplus \mathbb{R}, g_0)$ given by*

$$F_{\mu\nu} = n_\nu \left(\nabla^\Sigma_\mu N + K_{\mu\alpha} Y^\alpha \right) - n_\mu \left(\nabla^\Sigma_\nu N + K_{\nu\alpha} Y^\alpha \right) + f_{\mu\nu}, \tag{4.2.9}$$

where $f_{\mu\nu} = \nabla^\Sigma_{[\mu} Y_{\nu]}$.

In a spacetime setting it is well-known that for a non-trivial Killing vector $\vec{\xi}$, the Killing form cannot vanish on a fixed point. Let us show that the same happens in the KID setting.

Lemma 4.2.6 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a KID and $\mathfrak{p} \in \Sigma$ a fixed point, i.e. $N|_{\mathfrak{p}} = 0$ and $\vec{Y}|_{\mathfrak{p}} = 0$. If $F_{\mu\nu}|_{\mathfrak{p}} = 0$ then N and $\vec{Y}$ vanish identically on Σ.*

Proof. The aim is to obtain a suitable system of equations and show that, under the circumstances of the lemma, the solution must be identically zero. Decomposing $\nabla^\Sigma_i Y_j$ in symmetric and antisymmetric parts,

$$\nabla^\Sigma_i Y_j = -N K_{ij} + f_{ij}, \tag{4.2.10}$$

and inserting into (4.2.3) gives

$$\nabla^\Sigma_i \nabla^\Sigma_j N = N Q_{ij} - Y^l \nabla^\Sigma_l K_{ij} - K_{il} f_j{}^l - K_{jl} f_i{}^l, \tag{4.2.11}$$

where $Q_{ij} = R^{\Sigma}{}_{ij} + \mathrm{tr}_{\Sigma} K K_{ij} - \tau_{ij} + \frac{1}{2} g_{ij} (\mathrm{tr}_{\Sigma}\tau - \rho)$. In order to find an equation for $\nabla^{\Sigma}_{l} f_{ij}$, we take a derivative of (4.2.2) and write the three equations obtained by cyclic permutation. Adding two of them and subtracting the third one, we find,

$$\nabla^{\Sigma}_{l}\nabla^{\Sigma}_{i} Y_j = R^{\Sigma}{}_{klij} Y^k + \nabla^{\Sigma}_{j}(N K_{li}) - \nabla^{\Sigma}_{i}(N K_{lj}) - \nabla^{\Sigma}_{l}(N K_{ij}),$$

after using the Ricci and first Bianchi identities. Taking the antisymmetric part in i, j,

$$\nabla^{\Sigma}_{l} f_{ij} = R^{\Sigma}{}_{klij} Y^k + \nabla^{\Sigma}_{j} N K_{li} - \nabla^{\Sigma}_{i} N K_{lj} + N \nabla^{\Sigma}_{j} K_{li} - N \nabla^{\Sigma}_{i} K_{lj}. \qquad (4.2.12)$$

If $F_{\mu\nu}|_{\mathfrak{p}} = 0$, it follows that $f_{ij}|_{\mathfrak{p}} = 0$ and $\nabla^{\Sigma}_{i} N|_{\mathfrak{p}} = 0$. The equations given by (4.2.10), (4.2.11) and (4.2.12) is a system of PDE for the unknowns N, Y_i and f_{ij} written in normal form. It follows (see e.g. [55]) that the vanishing of N, $\nabla^{\Sigma}_{i} N$, Y_i and f_{ij} at one point implies its vanishing everywhere (recall that Σ is connected). ■

4.2.3 Canonical Form of Null two-forms

Let $F_{\mu\nu}$ be an arbitrary two-form on a spacetime $(M, g^{(4)})$. It is well-known that the only two non-trivial scalars that can be constructed from $F_{\mu\nu}$ are $I_1 = F_{\mu\nu} F^{\mu\nu}$ and $I_2 = F^{\star}_{\mu\nu} F^{\mu\nu}$, where $F^{\star}$ is the Hodge dual of F, defined by $F^{\star}_{\mu\nu} = \frac{1}{2} \eta^{(4)}_{\mu\nu\alpha\beta} F^{\alpha\beta}$, with $\eta^{(4)}_{\mu\nu\alpha\beta}$ being the volume form of $(M, g^{(4)})$. When both scalars vanish, the two-form is called *null*. Later on, we will encounter Killing forms which are null and we will exploit the following well-known algebraic decomposition which gives its **canonical form**, see e.g. [73] for a proof.

Lemma 4.2.7 *A null two-form $F_{\mu\nu}$ at a point $\mathfrak{p}$ can be decomposed as*

$$F_{\mu\nu}|_{\mathfrak{p}} = l_\mu w_\nu - l_\nu w_\mu|_{\mathfrak{p}}, \qquad (4.2.13)$$

where $\vec{l}|_{\mathfrak{p}}$ is a null vector and $\vec{w}|_{\mathfrak{p}}$ is spacelike and orthogonal to $\vec{l}|_{\mathfrak{p}}$.

4.3 Staticity of a KID

4.3.1 Static KID

To define a static KID we have to decompose the integrability equation $\xi_{[\mu} \nabla_\nu \xi_{\rho]} = 0$ according to (4.2.1). By taking the normal-tangent-tangent part (to Σ) and

the completely tangential part (the other components are identically zero by antisymmetry) we find

$$N\nabla^{\Sigma}_{[i}Y_{j]} + 2Y_{[i}\nabla^{\Sigma}_{j]}N + 2Y_{[i}K_{j]l}Y^{l} = 0, \tag{4.3.1}$$

$$Y_{[i}\nabla^{\Sigma}_{j}Y_{k]} = 0. \tag{4.3.2}$$

Since these expressions involve only objects on the KID, the following definition becomes natural.

Definition 4.3.1 *A KID* $(\Sigma, g, K; N, \vec{Y}, \tau)$ *satisfying (4.3.1) and (4.3.2) is called an* **integrable KID**.

Multiplying equation (4.3.1) by N and equation (4.3.2) by Y^k, adding them up and using equation (4.2.2), we get the following useful relation, valid everywhere on Σ,

$$\lambda\nabla^{\Sigma}_{[i}Y_{j]} + Y_{[i}\nabla^{\Sigma}_{j]}\lambda = 0. \tag{4.3.3}$$

If $\lambda > 0$ in some non-empty set of the KID, the Killing vector is timelike in some non-empty set of the spacetime. Hence

Definition 4.3.2 *A* **static KID** *is an integrable KID with* $\lambda > 0$ *in some non-empty set.*

4.3.2 Killing Form of a Static KID

In Subsection 4.2.3 we introduced the invariant scalars I_1 and I_2 for any two-form in a spacetime. In this section we find their explicit expressions for the Killing form of an integrable KID in the region $\{\lambda > 0\}$.

Although not necessary, we will pass to the Killing development (which is available in this case) since this simplifies the proofs. We start with a lemma concerning the integrability of the Killing vector in the Killing development.

Lemma 4.3.3 *The Killing vector field associated with the Killing development of an integrable KID is also integrable.*

Proof. Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be an integrable KID. Suppose the Killing development (4.2.4) of a suitable open set of Σ. Using $\vec{\xi} = \partial_t$ it follows

$$\boldsymbol{\xi}\wedge d\boldsymbol{\xi} = -\hat{\lambda}\partial_i\hat{Y}_j dt\wedge dx^i\wedge dx^j - \hat{Y}_i\partial_j\hat{\lambda} dt\wedge dx^i\wedge dx^j + \hat{Y}_i\partial_j\hat{Y}_k dx^i\wedge dx^j\wedge dx^k, \tag{4.3.4}$$

where $\hat{\lambda}$, $\hat{\mathbf{Y}}$ and $\hat{g}$ are defined in (4.2.5). Integrability of $\vec{\xi}$ follows directly from (4.3.2) and (4.3.3). ■

The following lemma gives the explicit expressions for I_1 and I_2.

Lemma 4.3.4 *The invariants of the Killing form in a static KID in the region* $\{\lambda > 0\}$ *read*

$$I_1 = -\frac{1}{2\lambda}\left(g^{ij} - \frac{Y^i Y^j}{N^2}\right)\nabla_i^\Sigma \lambda \nabla_j^\Sigma \lambda, \tag{4.3.5}$$

and

$$I_2 = 0. \tag{4.3.6}$$

Remark. By continuity $I_2|_{\partial^{top}\{\lambda>0\}} = 0$. □

Proof. Consider a static KID $(\Sigma, g, K; N, \vec{Y}, \tau)$ and let $\{\lambda > 0\}_0$ be a connected component of $\{\lambda > 0\}$. In $\{\lambda > 0\}_0$ we have necessarily $N \neq 0$, so we can construct the Killing development $(\{\lambda > 0\}_0, g^{(4)})$ and introduce the so-called Ernst one-form, as $\sigma_\mu = \nabla_\mu \lambda - i\omega_\mu$ where $\omega_\mu = \eta^{(4)}_{\mu\nu\alpha\beta}\xi^\nu \nabla^\alpha \xi^\beta$ is the twist of the Killing field ($\eta^{(4)}$ is the volume form of the Killing development). The Ernst one-form satisfies the identity (see e.g. [79]) $\sigma^\mu \sigma_\mu = -\lambda \left(F_{\mu\nu} + iF^\star_{\mu\nu}\right)\left(F^{\mu\nu} + iF^{\star\mu\nu}\right)$, which in the static case (i.e. $\omega_\mu = 0$) becomes $\nabla_\mu \lambda \nabla^\mu \lambda = -2\lambda \left(F_{\mu\nu}F^{\mu\nu} + iF_{\mu\nu}F^{\star\mu\nu}\right)$ where the identity $F_{\mu\nu}F^{\mu\nu} = -F^\star_{\mu\nu}F^{\star\mu\nu}$ has been used. The imaginary part immediately gives (4.3.6). The real part gives $I_1 = -\frac{1}{2\lambda}|\nabla\lambda|^2_{g^{(4)}}$. Taking coordinates $\{t, x^i\}$ adapted to the Killing field ∂_t, it follows from (4.2.5) that $|\nabla\lambda|^2_{g^{(4)}} = {g^{(4)}}^{ij}\partial_i \lambda \partial_j \lambda$. It is well-known (and easily checked) that the contravariant spatial components of $g^{(4)}$ are ${g^{(4)}}^{ij} = g^{ij} - \frac{Y^i Y^j}{N^2}$, where g^{ij} is the inverse of g_{ij} and (4.3.5) follows. ■

This lemma allows us to prove the following result on the value of I_1 on the set $\{\lambda > 0\}$.

Lemma 4.3.5 $I_1|_{\{\lambda>0\}} \leq 0$ *in a static KID.*

Proof. Let $\mathfrak{q} \in \{\lambda > 0\} \subset \Sigma$ and define the vector $\vec{\xi} \equiv N\vec{n} + \vec{Y}$ on the vector space $(V_\mathfrak{q}, g_0)$ introduced in Section 4.2.2. Since $\vec{\xi}$ is timelike at $\mathfrak{q}$, we can introduce its orthogonal projector $h_{\mu\nu} = {g_0}_{\mu\nu} + \frac{\xi_\mu \xi_\nu}{\lambda}$ which is obviously positive semi-definite. If we pull it back onto $T_\mathfrak{q}\Sigma$ we obtain a positive definite metric, called *orbit space metric*,

$$h_{ij} = g_{ij} + \frac{Y_i Y_j}{\lambda}. \tag{4.3.7}$$

It is immediate to check that the inverse of h_{ij} is precisely the term in brackets in (4.3.5). Consequently, $I_1|_\mathfrak{q} \leq 0$ follows. ■

Remark. By continuity $I_1|_{\partial^{top}\{\lambda>0\}} \leq 0$. □

Furthermore, for the fixed points on the closure of $\{\lambda > 0\}$ we have the following result. Notice that $\partial^{top}\{\lambda > 0\} \subset \overline{\{N \neq 0\}}$. Since the result involves points where N vanishes, we cannot rely on the Killing development for its proof and an argument directly on the initial data set is needed.

Lemma 4.3.6 *Let $\mathfrak{p} \in \overline{\{\lambda > 0\}}$ be a fixed point of a static KID, then $I_1|_{\mathfrak{p}} < 0$.*

Proof. From the previous lemma it follows that $I_1|_p \leq 0$. It only remains to show that $I_1|_{\mathfrak{p}}$ cannot be zero. We argue by contradiction. Assuming that $I_1|_{\mathfrak{p}} = 0$ and using $I_2|_{\mathfrak{p}} = 0$ by Lemma 4.3.4, it follows that $F_{\mu\nu}$ is null at $\mathfrak{p}$. Lemma 4.2.7 implies the existence of a null vector $\vec{l}$ and a spacelike vector $\vec{w}$ on $V_{\mathfrak{p}}$ such that (4.2.13) holds. Since $\vec{w}$ is defined up to an arbitrary additive vector proportional to $\vec{l}$, we can choose $\vec{w}$ normal to $\vec{n}$ without loss of generality. Decompose $\vec{l}$ as $\vec{l} = a\,(\vec{x} + \vec{n})$ with $x^\mu x_\mu = 1$. We know from Lemma 4.2.6 that $a \neq 0$ (otherwise $F_{\mu\nu}|_{\mathfrak{p}} = 0$ and $\{\lambda > 0\}$ would be empty). Expression (4.2.9) and the canonical form (4.2.13) yield

$$F_{\mu\nu}|_{\mathfrak{p}} = 2n_{[\nu}\nabla^{\Sigma}_{\mu]}N + \nabla^{\Sigma}_{[\mu}Y_{\nu]}|_{\mathfrak{p}} = 2a\left(x_{[\mu}w_{\nu]} + n_{[\mu}w_{\nu]}\right).$$

The purely tangential and normal-tangential components of this equation give, respectively

$$\nabla^{\Sigma}_{i}Y_{j}\big|_{\mathfrak{p}} = 2ax_{[i}w_{j]}, \quad \nabla^{\Sigma}_{i}N\big|_{\mathfrak{p}} = -aw_i, \tag{4.3.8}$$

where w_i is the projection of w_μ to $T_{\mathfrak{p}}\Sigma$. The Hessian of λ at $\mathfrak{p}$ is then

$$\begin{aligned} \nabla^{\Sigma}_{i}\nabla^{\Sigma}_{j}\lambda\big|_{\mathfrak{p}} &= 2(\nabla^{\Sigma}_{i}N\nabla^{\Sigma}_{j}N - \nabla^{\Sigma}_{i}Y^k\nabla^{\Sigma}_{j}Y_k)\big|_{\mathfrak{p}} \\ &= -2a^2w^kw_kx_ix_j, \end{aligned}$$

where we have used $x^ix_i = 1$ and $x^iw_i = 0$ (which follows from $\vec{w}$ being orthogonal to $\vec{l}$). This Hessian has therefore signature $\{-, 0, 0\}$. The Gromoll-Meyer splitting Lemma (see Appendix B) implies the existence of an open neighbourhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ and coordinates $\{x, z^A\}$ in $U_{\mathfrak{p}}$ such that $\mathfrak{p} = (x = 0, z^A = 0)$ and $\lambda = -\hat{a}^2x^2 + \zeta(z^A)$ where $\hat{a} > 0$ and ζ is a smooth function satisfying $\zeta\big|_{\mathfrak{p}} = 0$, $\nabla^{\Sigma}_{i}\zeta\big|_{\mathfrak{p}} = 0$ and $\nabla^{\Sigma}_{i}\nabla^{\Sigma}_{j}\zeta\big|_{\mathfrak{p}} = 0$. Since $\mathfrak{p} \in \partial^{top}\{\lambda > 0\}$, there exists a curve $\mu(s) = (x(s), z^A(s))$ in $U_{\mathfrak{p}} \cap \{\lambda > 0\}$, parametrized by $s \in (0, \epsilon)$ such that $\mu(s) \underset{s\to 0}{\longrightarrow} \mathfrak{p}$. Since $\lambda > 0$ on the curve we have $-\hat{a}^2x^2(s) + \zeta(z^A(s)) > 0$, which implies $\zeta(z^A(s)) > 0$. It follows that the curve $\gamma(s) \equiv \left(x(s) = \frac{1}{\hat{a}}\sqrt{\zeta(z^A(s))}, z^A(s)\right)$ (also parametrized by s) belongs to $\partial^{top}\{\lambda > 0\}$ and is composed by non-fixed points (because $\nabla^{\Sigma}_{i}\lambda\big|_{\gamma(s)} \neq 0$).

We can construct the Killing development (4.2.4) near this curve, which is a static spacetime (see Lemma 4.3.3). Applying Lemma 2.4.8 by Vishveshwara and Carter it follows that $\gamma(s)$ (which belongs to $\partial^{top}\{\lambda > 0\}$ and has $N \neq 0$) lies in an arc-connected component of a Killing prehorizon of the Killing development. Projecting equation (2.4.1), valid on a Killing prehorizon, onto Σ, we get the relation

$$\nabla_i^{\Sigma}\lambda\big|_{\gamma(s)} = 2\kappa Y_i\big|_{\gamma(s)}, \qquad (4.3.9)$$

where κ is the surface gravity of the prehorizon. Therefore, $\kappa\big|_{\gamma(s)} \neq 0$. Since $I_1 = -2\kappa^2$ (see e.g. equation (12.5.14) in [112]) and κ remains constant on $\gamma(s)$ (see Lemma 2.4.5), it follows, by continuity of I_1, that $I_1\big|_{\mathfrak{p}} = -2\kappa^2 < 0$. ■

4.3.3 Properties of $\partial^{top}\{\lambda > 0\}$ on a Static KID

In this subsection we will show that, under suitable conditions, the boundary of the region $\{\lambda > 0\}$ is a smooth surface. Our first result on the smoothness of $\partial^{top}\{\lambda > 0\}$ is the following.

Lemma 4.3.7 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a static KID and assume that the set $\mathcal{S} = \partial^{top}\{\lambda > 0\} \cap \{N \neq 0\}$ is non-empty. Then $\mathcal{S}$ is a smooth submanifold of Σ.*

Recall that in this thesis, a submanifold is, by definition, injectively immersed, but not necessarily embedded. Besides, it is worth to remark they are also not necessarily arc-connected.

Proof. Since $N|_{\mathcal{S}} \neq 0$, we can construct the Killing development (4.2.4) of a suitable neighbourhood of $\mathcal{S} \subset \Sigma$ satisfying $N \neq 0$ everywhere. Moreover, by Lemma 4.3.3, $\vec{\xi} = \partial_t$ is integrable. Applying Lemma 2.4.8 by Vishveshwara and Carter, it follows that the spacetime subset $\mathcal{N}_{\vec{\xi}} \equiv \partial^{top}\{\lambda > 0\} \cap \{\vec{\xi} \neq 0\}$ is a smooth null submanifold (in fact, a Killing prehorizon) of the Killing development and therefore transverse to Σ, which is spacelike. Thus, $\mathcal{S} = \Sigma \cap \mathcal{N}_{\vec{\xi}}$ is a smooth submanifold of Σ. ■

This lemma states that the boundary of $\{\lambda > 0\}$ is smooth on the set of non-fixed points. In fact, for the case of boundaries having at least one fixed point, an explicit defining function for this surface on the subset of non-fixed points can be given:

Lemma 4.3.8 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a static KID. If an arc-connected component of $\partial^{top}\{\lambda > 0\}$ contains at least one fixed point, then $\nabla^{\Sigma}_i \lambda \neq 0$ on all non-fixed points in that arc-connected component.*

Proof. Let V be the set of non-fixed points in one of the arc-connected components under consideration. This set is obviously open with at least one fixed point in its closure. Constructing the Killing development as before, we know that V belongs to a Killing prehorizon $\mathcal{H}_{\vec{\xi}}$. Projecting equation (2.4.1) onto Σ we get $\nabla^{\Sigma}_i \lambda\big|_{\mathcal{H}_{\vec{\xi}}\cap\Sigma} = 2\kappa Y_i\big|_{\mathcal{H}_{\vec{\xi}}\cap\Sigma}$. Since the surface gravity κ is constant on each arc-connected component of $\mathcal{H}_{\vec{\xi}}$ and $I_1 = -2\kappa^2$, Lemma 4.3.6 implies $\kappa\big|_V \neq 0$ and consequently $\nabla^{\Sigma}_i \lambda\big|_V \neq 0$. ■

Fixed points are more difficult to analyze. We first need a lemma on the structure of $\nabla^{\Sigma}_i N$ and f_{ij} on a fixed point.

Lemma 4.3.9 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a static KID and $\mathfrak{p} \in \partial^{top}\{\lambda > 0\}$ be a fixed point. Then*

$$\nabla^{\Sigma}_i N|_{\mathfrak{p}} \neq 0$$

and

$$f_{ij}|_{\mathfrak{p}} = \frac{b}{Q}\left(\nabla^{\Sigma}_i N X_j - \nabla^{\Sigma}_j N X_i\right)\Big|_{\mathfrak{p}} \tag{4.3.10}$$

where b is a constant, X_i is unit and orthogonal to $\nabla^{\Sigma}_i N|_{\mathfrak{p}}$ and $Q = +\sqrt{\nabla^{\Sigma}_i N \nabla^{\Sigma i} N}$.

Proof. From (4.2.9),

$$I_1 = F_{\mu\nu}F^{\mu\nu} = f_{ij}f^{ij} - 2\left(\nabla^{\Sigma}_i N + K_{ij}Y^j\right)\left(\nabla^{\Sigma^i} N + K^{ik}Y_k\right). \tag{4.3.11}$$

Hence, $\nabla^{\Sigma}_i N|_{\mathfrak{p}} \neq 0$ follows directly from $I_1|_{\mathfrak{p}} < 0$ (Lemma 4.3.6). For the second statement, let u_i be unit and satisfy $\nabla^{\Sigma}_i N = Q u_i$ in a suitable neighbourhood of $\mathfrak{p}$. Consider (4.3.1) in the region $N \neq 0$, which gives

$$f_{ij} = -2N^{-1}Y_{[i}\left(\nabla^{\Sigma}_{j]} N + K_{j]k}Y^k\right). \tag{4.3.12}$$

Since $|\vec{Y}|/N$ stays bounded in the region $\{\lambda > 0\}$, it follows that the second term tends to zero at the fixed point $\mathfrak{p}$. Thus, let $\vec{X}_1$ and $\vec{X}_2$ be any pair of vector fields orthogonal to $\vec{u}$. It follows by continuity that $f_{ij}X^i_1 X^j_2|_{\mathfrak{p}} = 0$. Hence for any orthonormal basis $\{\vec{u}, \vec{X}, \vec{Z}\}$ at $\mathfrak{p}$ it follows $f_{ij}X^i Z^j|_{\mathfrak{p}} = 0$ (because $\vec{X}$ and $\vec{Z}$ can be extended to a neighbourhood of $\mathfrak{p}$ while remaining orthogonal to $\vec{u}$).

Consequently, $f_{ij}|_{\mathfrak{p}} = (b/Q)(\nabla^\Sigma_i N X_j - \nabla^\Sigma_j N X_i) + (c/Q)(\nabla^\Sigma_i N Z_j - \nabla^\Sigma_j N Z_i)|_{\mathfrak{p}}$ for some constants b and c. A suitable rotation in the $\{\vec{X}, \vec{Z}\}$ plane allows us to set $c = 0$ and (4.3.10) follows. ■

As we will see next, a consequence of this lemma is that an open subset of fixed points in $\partial^{top}\{\lambda > 0\}$ is a smooth surface. In fact, we will prove that this surface is totally geodesic in (Σ, g) and that the pull-back of the second fundamental form K_{ij} vanishes there. This means from a spacetime perspective, i.e. when the initial data set is embedded into a spacetime, that this open set of fixed points is totally geodesic as a spacetime submanifold. This is of course well-known in the spacetime setting from Boyer's results [17], see also [67]. In our initial data context, however, the result must be proven from scratch as no Killing development is available at the fixed points.

Proposition 4.3.10 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a static KID and assume that the set $\partial^{top}\{\lambda > 0\}$ is non-empty. If $\mathcal{S} \subset \partial^{top}\{\lambda > 0\}$ is open and consists of fixed points, then $\mathcal{S}$ is a smooth surface. Moreover, the second fundamental form of $\mathcal{S}$ in (Σ, g) vanishes and $K_{AB}\big|_{\mathcal{S}} = 0$*

Proof. Consider a point $\mathfrak{p} \in \mathcal{S}$. We know from Lemma 4.3.9 that $\nabla^\Sigma_i N\big|_{\mathfrak{p}} \neq 0$. This means that there exists an open neighbourhood $U_{\mathfrak{p}}$ such that $\{N = \text{const}\} \cap U_{\mathfrak{p}}$ defines a foliation by smooth and connected surfaces, and moreover that $\nabla^\Sigma_i N \neq 0$ everywhere on $U_{\mathfrak{p}}$. Restricting $U_{\mathfrak{p}}$ if necessary we can assume that $\partial^{top}\{\lambda > 0\} \cap U_{\mathfrak{p}} = \mathcal{S} \cap U_{\mathfrak{p}}$ (because $\mathcal{S}$ is an open subset of $\partial^{top}\{\lambda > 0\}$). It is clear that $\mathcal{S} \cap U_{\mathfrak{p}} \subset \{N = 0\} \cap U_{\mathfrak{p}}$ (because N vanishes on a fixed point). We only need to prove that these two sets are in fact equal. Choose a continuous curve $\gamma : (-\epsilon, 0) \to \{\lambda > 0\} \cap U_{\mathfrak{p}}$ satisfying $\lim_{s\to 0}\gamma(s) = \mathfrak{p}$. Assume that there is a point $\mathfrak{q} \in \{N = 0\} \cap U_{\mathfrak{p}}$ not lying in $\partial^{top}\{\lambda > 0\}$. This means that there is an open neighbourhood $U_{\mathfrak{q}}$ of $\mathfrak{q}$ (which can be taken fully contained in $U_{\mathfrak{p}}$) which does not intersect $\{\lambda > 0\}$. Take a point $\mathfrak{r}$ in $U_{\mathfrak{q}}$ sufficiently close to $\mathfrak{q}$ so that $N\big|_{\mathfrak{r}}$ takes the same value as $N\big|_{\gamma(s_0)}$ for some $s_0 \in (-\epsilon, 0)$ (this point $\mathfrak{r}$ exists because $\nabla^\Sigma_i N\big|_{\mathfrak{q}} \neq 0$ and $N\big|_{\mathfrak{q}} = 0$). Since the surface $\{N = N\big|_{\mathfrak{r}}\} \cap U_{\mathfrak{p}}$ is connected and contains both $\mathfrak{r}$ and $\gamma(s_0)$, it follows that there is a path in $U_{\mathfrak{p}}$ with $N = N\big|_{\mathfrak{r}}$ constant and connecting these two points. This path must necessarily intersect $\partial^{top}\{\lambda > 0\}$ (recall that $\lambda\big|_{\gamma(s)} > 0$ for all s). But this contradicts the fact that $\partial^{top}\{\lambda > 0\} \cap U_{\mathfrak{p}} \subset \{N = 0\} \cap U_{\mathfrak{p}}$. Therefore, $\mathcal{S} \cap U_{\mathfrak{p}} = \{N = 0\} \cap U_{\mathfrak{p}}$, which proves that $\mathcal{S}$ is a smooth surface.

To prove the other statements, let us introduce local coordinates $\{u, x^A\}$ on Σ adapted to $\mathcal{S}$ so that $\mathcal{S} \equiv \{u = 0\}$ and let us prove that the linear term in a Taylor expansion for Y^i vanishes identically. Equivalently, we want to show that $u^j \nabla^\Sigma_j Y_i|_\mathcal{S} = 0$ for $\vec{u} = \partial_u$ (recall that on $\mathcal{S}$ we have $Y_i|_\mathcal{S} = 0$ and this covariant derivative coincides with the partial derivative). Note that $\nabla^\Sigma_i Y_j|_\mathcal{S} = f_{ij}$ (see (4.2.10)), so that $u^i u^j \nabla^\Sigma_i Y_j|_\mathcal{S} = 0$ being the contraction of a symmetric and an antisymmetric tensor. Moreover, for the tangential vectors $e^i_A = \partial_A$ we find $u^j e^i_A \nabla^\Sigma_i Y_j|_\mathcal{S} = u^j \partial_A Y_j = 0$ because Y_j vanishes all along $\mathcal{S}$. Consequently $u^i \partial_i Y_j|_\mathcal{S} = 0$. Hence, the Taylor expansion reads

$$\begin{aligned} N &= G(x^A)u + O(u^2), \\ Y_i &= O(u^2). \end{aligned} \tag{4.3.13}$$

Moreover, $G \neq 0$ everywhere on $\mathcal{S}$ because substituting this Taylor expansion in (4.3.5) and taking the limit $u \to 0$ gives $I_1|_\mathcal{S} = -2g^{uu}G^2(x^A)$ and we know that $I_1|_\mathcal{S} \neq 0$ from Lemma 4.3.6.

We can now prove that $\mathcal{S}$ is totally geodesic and that $K_{AB} = 0$. For the first, the Taylor expansion above gives

$$f_{ij}|_\mathcal{S} = 0 \tag{4.3.14}$$

and obviously N and $\vec{Y}$ also vanish on $\mathcal{S}$. Hence, from (4.2.11),

$$\nabla^\Sigma_i \nabla^\Sigma_j N|_\mathcal{S} = 0. \tag{4.3.15}$$

Since, by Lemma 4.3.9, $\nabla^\Sigma_i N|_\mathcal{S}$ is proportional to the unit normal to $\mathcal{S}$ and non-zero, then $\nabla^\Sigma_i \nabla^\Sigma_j N|_\mathcal{S} = 0$ is precisely the condition that $\mathcal{S}$ is totally geodesic. In order to prove $K_{AB}|_\mathcal{S} = 0$, we only need to substitute the Taylor expansion (4.3.13) in the AB components of (4.2.2). After dividing by u and taking the limit $u \to 0$, $K_{AB}|_\mathcal{S} = 0$ follows directly. ■

At this point, let us introduce a lemma on the constancy of I_1 on each arc-connected component of $\partial^{top}\{\lambda > 0\}$.

Lemma 4.3.11 *I_1 is constant on each arc-connected component of $\partial^{top}\{\lambda > 0\}$ in a static KID.*

Proof. For non-fixed points this is a consequence of the Vishveshwara-Carter Lemma (Lemma 2.4.8) and it has already been used several times before. For

an arc-connected open set $\mathcal{S}$ of fixed points, taking the derivative of equation (4.3.11) we get

$$\nabla^{\Sigma}_l I_1 = 2f^{ij}\nabla^{\Sigma}_l f_{ij} - 4(\nabla^{\Sigma}_l \nabla^{\Sigma}_i N + \nabla^{\Sigma}_l K_{ij} Y^j + K_{ij}\nabla^{\Sigma}_l Y^j)(\nabla^{\Sigma i} N + K^{ik} Y_k).$$

Then, using the facts that $f_{ij}\big|_{\mathcal{S}} = 0$ (equation (4.3.14)), $\nabla^{\Sigma}_i \nabla^{\Sigma}_j N\big|_{\mathcal{S}} = 0$ (equation (4.3.15)) and $\nabla^{\Sigma}_i Y_j = -NK_{ij} + f_{ij}$ (equation (4.2.10)), it is immediate to obtain that $\nabla^{\Sigma}_l I_1\big|_{\mathcal{S}} = 0$. Finally, continuity of I_1 leads to the result. ■

We have already proved that both the open sets of fixed points and the open sets of non-fixed points are smooth submanifolds. Unfortunately, when $\partial^{top}\{\lambda > 0\}$ contains fixed points not lying on open sets, this boundary is *not* a smooth submanifold in general. Consider as an example the Kruskal extension of the Schwarzschild black hole and choose one of the asymptotic regions where the static Killing field is timelike in the domain of outer communications. Its boundary consists of one half of the black hole event horizon, one half of the white hole event horizon and the bifurcation surface connecting both. Take an initial data set Σ that intersects the bifurcation surface transversally and let us denote by $\{\lambda > 0\}^{ext}$ the connected component of the subset $\{\lambda > 0\}$ within Σ contained in the chosen asymptotic region. The topological boundary $\partial^{top}\{\lambda > 0\}^{ext}$ is non-smooth because it has a corner on the bifurcation surface where the black hole event horizon and the white hole event horizon intersect (see example of Figure 4.1). We must therefore add some condition on $\partial^{top}\{\lambda > 0\}^{ext}$ in order to guarantee that this boundary does not intersect both a black and a white hole event horizon. In terms of the Killing vector, this requires that $\vec{Y}$ points only to one side of $\partial^{top}\{\lambda > 0\}^{ext}$. Lemma 4.3.8 suggests that the condition we need to impose is $Y^i \nabla^{\Sigma}_i \lambda\big|_{\partial^{top}\{\lambda>0\}^{ext}} \geq 0$ or $Y^i \nabla^{\Sigma}_i \lambda\big|_{\partial^{top}\{\lambda>0\}^{ext}} \leq 0$. This condition is in fact sufficient to show that $\partial^{top}\{\lambda > 0\}^{ext}$ is a smooth surface. Before giving the precise statement of this result (Proposition 4.3.14 below) we need to prove a lemma on the structure of λ near fixed points with $f_{ij} \neq 0$. For this, the following definition will be useful.

Definition 4.3.12 *A fixed point $\mathfrak{p} \in \partial^{top}\{\lambda > 0\}$ is called* **transverse** *if and only if $f_{ij}|_{\mathfrak{p}} \neq 0$ and* **non-transverse** *if and only if $f_{ij}|_{\mathfrak{p}} = 0$*

Lemma 4.3.13 *Let $\mathfrak{p} \in \partial^{top}\{\lambda > 0\}$ be a transverse fixed point. Then, there exists an open neighbourhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ and coordinates $\{x, y, z\}$ on $U_{\mathfrak{p}}$ such that $\lambda = \mu^2 x^2 - b^2 y^2$ for suitable constants $\mu > 0$ and $b \neq 0$.*

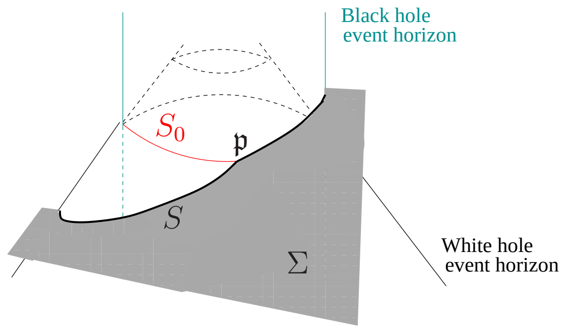


Figure 4.1: An example of non-smooth boundary $\mathcal{S} = \partial^{top}\{\lambda > 0\}$ in an initial data set Σ of Kruskal spacetime with one dimension suppressed. The region outside the cylinder and the cone corresponds to one asymptotic region of the Kruskal spacetime. The initial data set Σ intersects the bifurcation surface S_0 (in red). The shaded region corresponds to the intersection of Σ with the asymptotic region, and is in fact a connected component of the subset $\{\lambda > 0\} \subset \Sigma$. Its boundary is non-smooth at the point $\mathfrak{p}$ lying on the bifurcation surface.

Proof. From Lemma 4.3.9 we have $b \neq 0$. Squaring f_{ij} we get $f_{il}f_j{}^l|_{\mathfrak{p}} = b^2 \left(\frac{\nabla_i^\Sigma N \nabla_j^\Sigma N}{Q_0^2} + X_i X_j \right)\Big|_{\mathfrak{p}}$ and $f_{ij}f^{ij}|_{\mathfrak{p}} = 2b^2$, where $Q_0 = Q(\mathfrak{p})$. Being $\mathfrak{p}$ a fixed point, both λ and its gradient vanish at $\mathfrak{p}$ and we have a critical point. The Hessian of λ at $\mathfrak{p}$ is immediately computed to be

$$\begin{aligned} \nabla_i^\Sigma \nabla_j^\Sigma \lambda|_{\mathfrak{p}} &= 2\nabla_i^\Sigma N \nabla_j^\Sigma N - 2 f_{il} f_j{}^l\big|_{\mathfrak{p}} \\ &= \frac{2\left(Q_0^2 - b^2\right)}{Q_0^2} \nabla_i^\Sigma N \nabla_j^\Sigma N - 2b^2 X_i X_j \Big|_{\mathfrak{p}} . \end{aligned} \tag{4.3.16}$$

At a fixed point we have $I_1|_{\mathfrak{p}} = f_{ij}f^{ij} - 2\nabla_i^\Sigma N \nabla^{\Sigma i} N|_{\mathfrak{p}} = 2(b^2 - Q_0^2) < 0$ (Lemma 4.3.6). Let us define $\mu > 0$ by $\mu^2 = Q_0^2 - b^2$. The rank of the Hessian is therefore two and the signature is $(+,-,0)$. The Gromoll-Meyer splitting Lemma (see Appendix B) implies the existence of coordinates $\{x, y, z\}$ in a suitable neighbourhood $U'_{\mathfrak{p}}$ of $\mathfrak{p}$ such that $\mathfrak{p} = \{x = 0, y = 0, z = 0\}$ and $\lambda = \mu^2 x^2 - b^2 y^2 + h(z)$ on $U'_{\mathfrak{p}}$. The function $h(z)$ is smooth and satisfies $h(0) = h'(0) = h''(0) = 0$, where prime stands for derivative with respect to z. Moreover, evaluating the Hessian of λ at $\mathfrak{p}$ and comparing with (4.3.16) we have $dx|_{\mathfrak{p}} = Q_0^{-1} dN|_{\mathfrak{p}}$ and $dy|_{\mathfrak{p}} = \boldsymbol{X}$. This implies $N = Q_0 x + O(2)$. Moreover, since $\nabla_i^\Sigma Y_j|_{\mathfrak{p}} = f_{ij}|_{\mathfrak{p}} = b(dx \otimes dy - dy \otimes dx)_{ij}|_{\mathfrak{p}}$ we conclude $Y_x = -by + O(2)$, $Y_y = bx + O(2)$, $Y_z = O(2)$. On the surface $\{z = 0\}$, the set of points where λ vanishes is given by the two lines $x = x_+(y) \equiv b\mu^{-1} y$

and $x = x_-(y) \equiv -b\mu^{-1}y$. Computing the gradient of λ on these curves we find

$$d\lambda|_{(x=x_\pm(y),z=0)} = \pm 2\mu b y dx - 2b^2 y dy. \qquad (4.3.17)$$

On the other hand, the Taylor expansion above for $\boldsymbol{Y}$ gives

$$\boldsymbol{Y}|_{(x=x_\pm(y),z=0)} = -bydx \pm \frac{b^2}{\mu} ydy + O(2). \qquad (4.3.18)$$

Let $\mathcal{S}$ be the arc-connected component of $\partial^{top}\{\lambda > 0\}$ containing $\mathfrak{p}$. On all non-fixed points in $\mathcal{S}$ we have $d\lambda = 2\kappa \boldsymbol{Y}$, with $\kappa^2 = -I_1/2$. Comparing (4.3.17) with (4.3.18) yields $\kappa = -\mu$ on the branch $x = x_+(y)$ and $\kappa = +\mu$ on the branch $x = x_-(y)$ (this is in agreement with $I_1 = -2\kappa^2 = -2\mu^2$ at every point in $\mathcal{S}$). We already know that κ must remain constant on each arc-connected component of $\mathcal{S} \setminus F$, where $F = \{\mathfrak{p} \in \mathcal{S}, \mathfrak{p} \text{ fixed point}\}$. Let us show that this implies $h(z) = 0$ on $U'_{\mathfrak{p}}$. First, we notice that the set of fixed points on $\mathcal{S}$ are precisely those where $\lambda = 0$ and $d\lambda = 0$ (this is because in Lemma 4.3.8 we have shown that $d\lambda \neq 0$ on every non-fixed point of any arc-connected component of $\partial^{top}\{\lambda > 0\}$ containing at least one fixed point). From the expression $\lambda = \mu^2 x^2 - b^2 y^2 + h(z)$, this implies that the fixed points in $U'_{\mathfrak{p}}$ are those satisfying $\{x = 0, y = 0, h(z) = 0, h'(z) = 0\}$. Assume that there is no neighbourhood $(-\epsilon, \epsilon)$ where h vanishes identically. Then, there exists a sequence $z_n \to 0$ satisfying $h(z_n) \neq 0$. There must exist a subsequence (still denoted by $\{z_n\}$) satisfying either $h(z_n) > 0$, $\forall n \in \mathbb{N}$ or $h(z_n) < 0$, $\forall n \in \mathbb{N}$. The two cases are similar, so we only consider $h(z_n) = -a_n^2 < 0$. The set of points with $\lambda = 0$ in the surface $\{z = z_n\}$ are given by $x = \pm\mu^{-1}\sqrt{b^2 y^2 + a_n^2}$. It follows that the points $\{\lambda = 0\} \cap \{z = z_n\}$ in the quadrant $\{x > 0, y > 0\}$ lie in the same arc-connected component as the points $\{\lambda = 0\} \cap \{z = z_n\}$ lying in the quadrant $\{x > 0, y < 0\}$. Since z_n converges to zero, it follows that the points $\{x = x_+(y), y > 0, z = 0\}$ lie in the same arc-connected component of $\mathcal{S} \setminus F$ than the points $\{x = x_-(y), y < 0, z = 0\}$. However, this is impossible because κ (which is constant on $\mathcal{S} \setminus F$) takes opposite values on the branch $x = x_+(y)$ and on the branch $x = x_-(y)$. This gives a contradiction, and so there must exist a neighbourhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ where $h(z) = 0$. ■

Now, we are ready to prove a smoothness result for $\partial^{top}\{\lambda > 0\}$.

Proposition 4.3.14 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a static KID and consider a connected component $\{\lambda > 0\}_0$ of $\{\lambda > 0\}$. If $Y^i \nabla_i^\Sigma \lambda \geq 0$ or $Y^i \nabla_i^\Sigma \lambda \leq 0$ on an arc-connected component $\mathcal{S}$ of $\partial^{top}\{\lambda > 0\}_0$, then $\mathcal{S}$ is a smooth submanifold (i.e. injectively immersed) of Σ.*

Proof. If there are no fixed points in $\mathcal{S}$, the result follows from Lemma 4.3.7. Let us therefore assume that there is at least one fixed point $\mathfrak{p} \in \mathcal{S}$. The idea of the proof proceeds in three stages. The first stage will consist in showing that $Y^i \nabla^\Sigma_i \lambda \geq 0$ (or $Y^i \nabla^\Sigma_i \lambda \leq 0$) forces all fixed points in $\mathcal{S}$ to be non-transverse. The second one consists in proving that, in a neighbourhood of a non-transverse fixed point, $\mathcal{S}$ is a C^1 submanifold. In the third and final stage we prove that $\mathcal{S}$ is, in fact, C^∞.

Stage 1. We argue by contradiction. Assume the fixed point $\mathfrak{p}$ is transverse. Lemma 4.3.13 implies that either $\{\lambda > 0\}_0 \cap U_\mathfrak{p} = \{x > \frac{|b||y|}{\mu}\}$ or $\{\lambda > 0\}_0 \cap U_\mathfrak{p} = \{x < -\frac{|b||y|}{\mu}\}$. We treat the first case (the other is similar). The boundary of $\{\lambda > 0\}_0 \cap U_\mathfrak{p}$ is connected and given by $x = x_+(y)$ for $y > 0$ and $x = x_-(y)$ for $y < 0$. Using $d\lambda = 2\kappa \boldsymbol{Y}$ on this boundary, it follows $Y^i \nabla^\Sigma_i \lambda = 2\kappa Y_i Y^i$. But κ has different signs on the branch $x = x_+(y)$ and on the branch $x = x_-(y)$, so $Y^i \nabla^\Sigma_i \lambda$ also changes sign, against hypothesis. Hence $\mathfrak{p}$ must be a non-transverse fixed point.

Stage 2. Let us show that there exists a neighbourhood of $\mathfrak{p}$ where $\mathcal{S}$ is C^1. Being $\mathfrak{p}$ non-transverse, we have $f_{ij}|_\mathfrak{p} = 0$ and, consequently, the Hessian of λ reads

$$\nabla^\Sigma_i \nabla^\Sigma_j \lambda|_\mathfrak{p} = 2\nabla^\Sigma_i N \nabla^\Sigma_j N|_\mathfrak{p}, \tag{4.3.19}$$

which has signature $\{+, 0, 0\}$. Similarly as in Lemma 4.3.6, the Gromoll-Meyer splitting Lemma (see Appendix B) implies the existence of an open neighbourhood $U_\mathfrak{p}$ of $\mathfrak{p}$ and coordinates $\{x, z^A\}$ in $U_\mathfrak{p}$ such that $\mathfrak{p} = \{x = 0, z^A = 0\}$ and $\lambda = Q_0^2 x^2 - \zeta(z)$, where ζ is a smooth function satisfying $\zeta|_\mathfrak{p} = 0$, $\nabla^\Sigma_i \zeta|_\mathfrak{p} = 0$ and $\nabla^\Sigma_i \nabla^\Sigma_j \zeta|_\mathfrak{p} = 0$, and Q_0 is a positive constant. Moreover, evaluating the Hessian of $\lambda = Q_0^2 x^2 - \zeta(z)$ and comparing with (4.3.19) gives $dx|_\mathfrak{p} = Q_0^{-1} dN|_\mathfrak{p}$.

Let us first show that there exists a neighbourhood $V_\mathfrak{p}$ of $\mathfrak{p}$ where $\zeta \geq 0$. The surfaces $\{N = 0\}$ and $\{x = 0\}$ are tangent at $\mathfrak{p}$. This implies that there exists a neighbourhood $V_\mathfrak{p}$ of $\mathfrak{p}$ in Σ such that the integral lines of ∂_x are transverse to $\{N = 0\}$. Assume $\zeta(z) < 0$ on any of these integral lines. If follows that $\lambda = Q_0^2 x^2 - \zeta$ is positive everywhere on this line. But at the intersection with $\{N = 0\}$ we have $\lambda = N^2 - Y^i Y_i = -Y^i Y_i \leq 0$. This gives a contradiction and hence $\zeta(z) \geq 0$ in $V_\mathfrak{p}$ as claimed.

The set of points $\{\lambda > 0\} \cap V_\mathfrak{p}$ is given by the union of two disjoint connected sets namely $W_+ \equiv \{x > +\frac{\sqrt{\zeta}}{Q_0}\}$ and $W_- \equiv \{x < -\frac{\sqrt{\zeta}}{Q_0}\}$. On a connected component of $\{\lambda > 0\}$ (in particular on $\{\lambda > 0\}_0$) we have that $N = \sqrt{\lambda + Y^i Y_i}$ must be either everywhere positive or everywhere negative. On the other hand, for $\delta > 0$ small enough $N|_{(x=\delta, z^A=0)}$ must have different sign than $N|_{(x=-\delta, z^A=0)}$ (this

is because $\partial_x N|_{\mathfrak{p}} = dN(\partial_x)|_{\mathfrak{p}} = Q_0 dx(\partial_x)|_{\mathfrak{p}} > 0$). It follows that either $\{\lambda > 0\}_0 \cap V_{\mathfrak{p}} = W_+$ (if $N > 0$ in $\{\lambda > 0\}_0$) or $\{\lambda > 0\}_0 \cap V_{\mathfrak{p}} = W_-$ (if $N < 0$ in $\{\lambda > 0\}_0$). Consequently, $\mathcal{S}$ is locally defined by $x = \frac{\epsilon\sqrt{\zeta}}{Q_0}$, where ϵ is the sign of N in $\{\lambda > 0\}_0$. Now, we need to prove that $+\sqrt{\zeta}$ is C^1. This requires studying the behavior of ζ at points where it vanishes.

The set of fixed points $\mathfrak{p}' \in V_{\mathfrak{p}}$ is given by $\{x = 0, \zeta(z) = 0\}$ (this is a consequence of the fact that fixed points in $\mathcal{S}$ are characterized by the equations $\lambda = 0$ and $d\lambda = 0$, or equivalently $x = 0$, $\zeta = 0$, $d\zeta = 0$. Since, for non-negative functions, $\zeta = 0$ implies $d\zeta = 0$ the statement above follows). The Hessian of λ on any fixed point $\mathfrak{p}' \subset V_{\mathfrak{p}}$ reads $\nabla^\Sigma_i \nabla^\Sigma_j \lambda|_{\mathfrak{p}'} = 2Q_0^2 (dx \otimes dx)_{ij} - \nabla^\Sigma_i \nabla^\Sigma_j \zeta|_{\mathfrak{p}'}$. Since $\mathfrak{p}'$ must be a non-transverse fixed point, we have $\nabla^\Sigma_i Y_j|_{\mathfrak{p}'} = f_{ij}|_{\mathfrak{p}'} = 0$ and hence $\nabla^\Sigma_i \nabla^\Sigma_j \lambda|_{\mathfrak{p}'} = 2\nabla^\Sigma_i N \nabla^\Sigma_j N|_{\mathfrak{p}'}$ which has rank 1. Consequently, $\nabla^\Sigma_i \nabla^\Sigma_j \zeta|_{\mathfrak{p}'} = 0$. So, at all points where ζ vanishes we not only have $d\zeta = 0$ but also $\nabla^\Sigma_i \nabla^\Sigma_j \zeta = 0$. We can now apply a theorem by Glaeser (see Appendix B) to conclude that the positive square root $u \equiv \frac{+\sqrt{\zeta}}{Q_0}$ is C^1, as claimed.

Stage 3. Finally, we will prove that $\mathcal{S}$ is, in fact, C^∞ in a neighbourhood of $\mathfrak{p}$ (we already know that $\mathcal{S}$ is smooth at non-fixed points) This is equivalent to proving that the function $x = \epsilon u(z)$ is C^∞. Since $u = \frac{+\sqrt{\zeta}}{Q_0}$ and $\zeta \geq 0$, it follows that u is smooth at any point where $u > 0$. The proof will proceed in two steps. In the first step we will show that u is C^2 at points where u vanishes and then, we will improve this to C^∞. Let us start with the C^2 statement. At points where $u \neq 0$, we have $Y_i|_{(x=\epsilon u(z), z^A)} = \frac{1}{2\kappa} \nabla^\Sigma_i \lambda|_{(x=\epsilon u(z), z^A)}$. Hence Y_i is non-zero and orthogonal to $\mathcal{S}$ on such points. Pulling back equation $\nabla^\Sigma_i Y_j + \nabla^\Sigma_j Y_i + 2NK_{ij} = 0$ onto $\mathcal{S} \cap \{x \neq 0\}$, we get

$$\kappa_{AB} + \epsilon\sigma K_{AB} = 0, \qquad (4.3.20)$$

where σ is the sign of κ , K_{AB} is the pull-back of K_{ij} on the surface $\{x = \epsilon u(z)\}$ and κ_{AB} is the second fundamental form of this surface with respect to the unit normal pointing inside $\{\lambda > 0\}_0$. By assumption $Y^i \nabla^\Sigma_i \lambda$ has constant sign on $\mathcal{S}$. This implies that σ is either everywhere $+1$ or everywhere -1. So, the graph $x = \epsilon u(z)$ satisfies the set of equations $\kappa_{AB} + \epsilon\sigma K_{AB} = 0$ on the open set $\{z^A; u(z) > 0\} \subset \mathbb{R}^2$. In the local coordinates $\{z^A\}$ these equations takes the form

$$-\partial_A \partial_B u(z) + \chi_{AB}(u(z), \partial_C u(z), z) = 0 \qquad (4.3.21)$$

where χ is a smooth function of its arguments which satisfies $\chi_{AB}(u = 0, \partial_C u = 0, z) = \epsilon \hat{\kappa}_{AB}(z) + \sigma \hat{K}_{AB}(z)$, where $\hat{\kappa}_{AB}$ is the second fundamental form of the surface $\{x = 0\}$ (with respect to the outer normal pointing towards $\{x > 0\}$) at

the point with coordinates $\{z^A\}$ and $\hat{K}_{AB}$ is the pull-back of K_{ij} on this surface at the same point. Take a fixed point $\mathfrak{p}' \in \mathcal{S}$ not lying within an open set of fixed points (if $\mathfrak{p}'$ lies on an open set of fixed points we have $u \equiv 0$ on the open set and the statement that u is C^∞ is trivial). It follows that $\mathfrak{p}' \in \{x = 0\}$ and that the coordinates z_0^A of $\mathfrak{p}'$ satisfy $z_0^A \in \partial^{top}\{z^A; u(z) > 0\} \subset \mathbb{R}^2$. By stage 2 of the proof, the function $u(z)$ is C^1 everywhere and its gradient vanishes wherever u vanishes. It follows that $u\big|_{z_0^A} = \partial_B u\big|_{z_0^A} = 0$. Being u continuously differentiable, it follows that the term χ_{AB} in (4.3.21) is C^0 as a function of z^C and therefore admits a limit at z_0^C. It follows that $\partial_A\partial_B u$ also has a well-defined limit at z_0^C, and in fact this limit satisfies

$$\partial_A\partial_B u\big|_{z_0^C} = \hat{\kappa}_{AB}\big|_{z_0^C} + \epsilon\sigma\hat{K}_{AB}\big|_{z_0^C}.$$

This shows that u is in fact C^2 everywhere. But taking the trace of $\kappa_{AB} + \epsilon\sigma K_{AB} = 0$, we get $p + \epsilon\sigma q = 0$, where p is the mean curvature of $\mathcal{S}$ and q is the trace of the pull-back of K_{ij} on $\mathcal{S}$. This is an elliptic equation in the coordinates $\{z^A\}$ (see e.g. [3]), so C^2 solutions are smooth as a consequence of elliptic regularity [58]. Thus, the function $u(z)$ is C^∞. ■

Knowing that this submanifold is differentiable, our next aim is to show that, under suitable circumstances it has vanishing outer null expansion. This is the content of our next proposition.

Proposition 4.3.15 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be a static KID and consider a connected component $\{\lambda > 0\}_0$ of $\{\lambda > 0\}$ with non-empty topological boundary. Let $\mathcal{S}$ be an arc-connected component $\partial^{top}\{\lambda > 0\}_0$ and assume*

(i) $NY^i\nabla_i^\Sigma\lambda|_{\mathcal{S}} \geq 0$ if $\mathcal{S}$ contains at least one fixed point.

(ii) $NY^i m_i|_{\mathcal{S}} \geq 0$ if $\mathcal{S}$ contains no fixed point, where $\vec{m}$ is the unit normal pointing towards $\{\lambda > 0\}_0$.

Then $\mathcal{S}$ is a smooth submanifold (i.e. injectively immersed) with $\theta^+ = 0$ provided the outer direction is defined as the one pointing towards $\{\lambda > 0\}_0$. Moreover, if $I_1 \neq 0$ in $\mathcal{S}$, then $\mathcal{S}$ is embedded.

Remark. If the inequalities in (i) and (ii) are reversed, then $\mathcal{S}$ has $\theta^- = 0$. □

Proof. Consider first the case when $\mathcal{S}$ has at least one fixed point. Since, on $\mathcal{S}$, N cannot change sign and vanishes only if $\vec{Y}$ also vanishes, the hypothesis

$NY^i\nabla_i^\Sigma\lambda|_{\mathcal{S}} \geq 0$ implies either $Y^i\nabla_i^\Sigma\lambda|_{\mathcal{S}} \geq 0$ or $Y^i\nabla_i^\Sigma\lambda|_{\mathcal{S}} \leq 0$ and, therefore, Proposition 4.3.14 shows that $\mathcal{S}$ is a smooth submanifold. Let $\vec{m}$ be the unit normal pointing towards $\{\lambda > 0\}_0$ and p the corresponding mean curvature. We have to show that $\theta^+ = p + \gamma^{AB}K_{AB}$ (see equation (2.2.8)) vanishes. Open sets of fixed points are immediately covered by Proposition 4.3.10 because this set is then totally geodesic and $K_{AB} = 0$, so that both null expansions vanish.

On the subset $V \subset \mathcal{S}$ of non-fixed points we have $Y_i\big|_V = \frac{1}{2\kappa}\nabla_i^\Sigma\lambda\big|_V$ (see equation 4.3.9) and, therefore, $Y_i\big|_V = |N|\text{sign}(\kappa)m_i\big|_V$. The condition $NY^i\nabla_i^\Sigma\lambda \geq 0$ imposes $\text{sign}(N)\text{sign}(\kappa) = 1$ or, in the notation of the proof of Proposition 4.3.14, $\epsilon\sigma = 1$. Equation $p + q = 0$ follows directly from (4.3.20) after taking the trace.

For the case (ii), we know that $\mathcal{S}$ is smooth from Lemma 4.3.7 and, hence, $\vec{m}$ exists (this shows in particular that hypothesis (ii) is well-defined). Since $\mathcal{S}$ lies in a Killing prehorizon in the Killing development of the KID, it follows that $\vec{\xi}$ is orthogonal to $\mathcal{S}$ and hence that $\vec{Y}$ is normal to $\mathcal{S}$ in Σ. Since $\vec{Y}^2 = N^2$ on $\mathcal{S}$ it follows $\vec{Y}|_{\mathcal{S}} = N\vec{m}|_{\mathcal{S}}$ and the same argument applies to conclude $\theta^+ = 0$.

To show that $\mathcal{S}$ is embedded if $I_1|_{\mathcal{S}} \neq 0$, consider a point $\mathfrak{p} \in \mathcal{S}$. if $\mathfrak{p}$ is a non-fixed point, we know that $\nabla_i^\Sigma\lambda\big|_{\mathfrak{p}} \neq 0$ and hence λ is a defining function for $\mathcal{S}$ in a neighbourhood of $\mathfrak{p}$. This immediately implies that $\mathcal{S}$ is embedded in a neighbourhood of $\mathfrak{p}$. When $\mathfrak{p}$ is a fixed point, we have shown in the proof of Proposition 4.3.14 that there exists an open neighbourhood $V_{\mathfrak{p}}$ of $\mathfrak{p}$ such that, in suitable coordinates, $\overline{\{\lambda > 0\}} \cap V_{\mathfrak{p}} = \{x \geq u(z)\}$ or $\overline{\{\lambda > 0\}} \cap V_{\mathfrak{p}} = \{x \leq -u(z)\}$ for a non-negative smooth function $u(z)$. It is clear that the arc-connected component $\mathcal{S}$ is defined locally by $x = u(z)$ or $x = -u(z)$ and hence it is embedded. ■

4.4 The confinement result

Now, we are ready to state and prove our confinement result. For simplicity, it will be formulated as a confinement result for outer trapped surfaces instead of weakly outer trapped surfaces. However, except for a singular situation, it can be immediately extended to weakly outer trapped surfaces (see Remark 1 after the proof).

Theorem 4.4.1 *Consider a static KID $(\Sigma, g, K; N, \vec{Y}, \tau)$ satisfying the NEC and possessing a barrier S_b with interior Ω_b (see Definition 2.2.25) which is outer untrapped and such that such that $\lambda\big|_{S_b} > 0$. Let $\{\lambda > 0\}^{ext}$ be the connected*

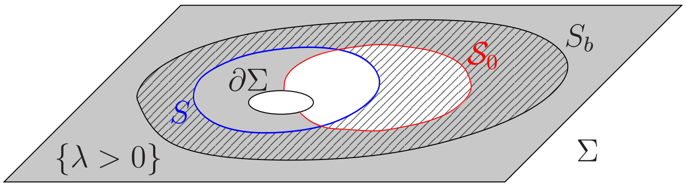


Figure 4.2: Theorem 4.4.1 forbids the existence of an outer trapped surface S like the one in the figure (in blue). The striped area corresponds to the exterior of S in Ω_b and the shaded area corresponds to the set $\{\lambda > 0\}^{\text{ext}}$ whose boundary is $\mathcal{S}_0$ (in red). Note that $\mathcal{S}_0$ may intersect $\partial\Sigma$.

component of $\{\lambda > 0\}$ containing S_b. Assume that every arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ with $I_1 = 0$ is topologically closed and

1. *$NY^i\nabla^{\Sigma}_i\lambda \geq 0$ in each arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ containing at least one fixed point.*

2. *$NY^i m_i \geq 0$ in each arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ which contains no fixed points, where $\vec{m}$ is the unit normal pointing towards $\{\lambda > 0\}^{ext}$.*

Consider any surface S which is bounding with respect to S_b. If S is outer trapped then it does not intersect $\{\lambda > 0\}^{ext}$.

Proof. We argue by contradiction. Let S be an outer trapped surface which is bounding with respect to S_b, satisfies the hypotheses of the theorem and intersects $\{\lambda > 0\}^{\text{ext}}$. By definition of bounding, there exists a compact manifold $\tilde{\Sigma}$ whose boundary is the disjoint union of the outer untrapped surface S_b and the outer trapped surface S. We work on $\tilde{\Sigma}$ from now on. The Andersson and Metzger Theorem 2.2.31 implies that the topological boundary of the weakly outer trapped region $\partial^{top}T^+$ in $\tilde{\Sigma}$ is a stable MOTS which is bounding with respect to S_b. We first show that $\partial^{top}T^+$ necessarily intersects $\{\lambda > 0\}^{\text{ext}}$. Indeed, consider a point $\mathfrak{r} \in S$ with $\lambda|_{\mathfrak{r}} > 0$ (this point exists by hypothesis) and consider a path from $\mathfrak{r}$ to S_b fully contained in $\{\lambda > 0\}^{\text{ext}}$ (this path exists because $\{\lambda > 0\}^{\text{ext}}$ is connected). Since $\mathfrak{r} \in T^+$ it follows that this path must intersect $\partial^{top}T^+$ as claimed. Furthermore, due to the maximum principle for MOTS (see Proposition B.7), $\partial^{top}T^+$ lies entirely in the exterior of S in Ω_b (here is where we use the hypothesis of S being outer trapped instead of merely being weakly outer trapped).

Let us suppose for a moment that $\partial^{top}T^+ \subset \overline{\{\lambda > 0\}^{\text{ext}}}$. Then the Killing vector $N\vec{n} + \vec{Y}$ is causal everywhere on $\partial^{top}T^+$, either future or past directed, and timelike somewhere on $\partial^{top}T^+$. Since $\partial^{top}T^+$ intersects $\{\lambda > 0\}^{\text{ext}}$, there must be non-fixed points on $\partial^{top}T^+$. If all points in $\partial^{top}T^+$ are non-fixed, then we can construct the Killing development and Theorem 3.4.9 can be applied at once giving a contradiction (note that $\partial^{top}T^+$ is necessarily a locally outermost MOTS). When $\partial^{top}T^+$ has fixed points we cannot construct the Killing development everywhere. However, let $V \subset \partial^{top}T^+$ be a connected component of the set of non-fixed points in $\partial^{top}T^+$ satisfying $V \cap \{\lambda > 0\} \neq \emptyset$ (this V exists because $\lambda > 0$ somewhere on $\partial^{top}T^+$). Then, the Killing development still exists in an open neighbourhood of V. In this portion we can repeat the geometrical construction which allowed us to prove Theorem 3.4.9 and define a surface S' by moving V a small, but finite amount τ along $\vec{\xi}$ to the past and back to Σ along the outer null geodesics. Since N and $\vec{Y}$ are smooth and approach zero at $\partial^{top}V$ it follows that S' and the set of fixed points in $\partial^{top}T^+$ join smoothly and therefore define a closed surface S''. Clearly, S'' is weakly outer trapped and lies, at least partially, in the exterior of $\partial^{top}T^+$, which is impossible.

Until now, we have essentially applied the ideas of Theorem 3.4.9. When $\partial^{top}T^+ \not\subset \overline{\{\lambda > 0\}^{\text{ext}}}$ new methods are required. However, the general strategy is still to construct a weakly outer trapped surface outside $\partial^{top}T^+$ in $\tilde{\Sigma}$.

First of all, every arc-connected component $\mathcal{S}_i$ of $\partial^{top}\{\lambda > 0\}^{ext}$ with $I_1 \neq 0$ is embedded, as proven in Proposition 4.3.15. For an arc-connected component $\mathcal{S}_d$ with $I_1 = 0$ we note that, since no point on this set is a fixed point, it follows that there exists an open neighbourhood U of $\mathcal{S}_d$ containing no fixed points. Thus, the vector field $\vec{Y}$ is nowhere zero on U. Staticity of the KID implies that $\boldsymbol{Y}$ is integrable (see (4.3.2)). It follows by the Fröbenius theorem that U can be foliated by maximal, injectively immersed submanifolds orthogonal to $\vec{Y}$. $\mathcal{S}_d$ is clearly one of the leaves of this foliation because $\vec{Y}$ is orthogonal to $\mathcal{S}_d$ everywhere. By assumption, $\mathcal{S}_d$ is topologically closed. Now, we can invoke a result on the theory of foliations that states that any topologically closed leaf in a foliation is necessarily embedded (see e.g. Theorem 5 in page 51 of [91]). Thus, each $\mathcal{S}_i$ is an embedded submanifold of $\tilde{\Sigma}$. Since we know that $\partial^{top}T^+$ intersects $\{\lambda > 0\}^{ext}$ and we are assuming that $\partial^{top}T^+ \not\subset \overline{\{\lambda > 0\}^{\text{ext}}}$, it follows that at least one of the arc-connected components $\{\mathcal{S}_i\}$, say $\mathcal{S}_0$, must intersect both the interior and the exterior of $\partial^{top}T^+$. In Proposition 4.3.15 we have also shown that $\mathcal{S}_0$ has $\theta^+ = 0$ with respect to the direction pointing towards $\{\lambda > 0\}^{\text{ext}}$.

Thus, we have two intersecting surfaces $\partial^{top}T^+$ and $\mathcal{S}_0$ which satisfy $\theta^+ = 0$. Moreover, $\partial^{top}T^+$ is a stable MOTS. The idea is to use Lemma 3.5.1 by Kriele and Hayward to construct a weakly outer trapped surface $\hat{S}$ outside both $\partial^{top}T^+$ and $\mathcal{S}_0$ and which is bounding with respect to S_b. However, Lemma 3.5.1 can be applied directly only when both surfaces $\partial^{top}T^+$ and $\mathcal{S}_0$ intersect transversally in a curve and this need not happen for $\mathcal{S}_0$ and $\partial^{top}T^+$. To address this issue we use a technique developed by Andersson and Metzger in their proof of Theorems 5.1 and 7.6 in [4].

The idea is to use Sard Lemma (see Appendix B) in order to find a weakly outer trapped surface $\tilde{S}$ as close to $\partial^{top}T^+$ as desired which does intersect $\mathcal{S}_0$ transversally. Then, the Kriele and Hayward smoothing procedure applied to $\tilde{S}$ and $\mathcal{S}_0$ gives a weakly outer trapped surface penetrating $\tilde{\Sigma} \setminus T^+$, which is simply impossible.

So, it only remains to prove the existence of $\tilde{S}$.

Recall that $\partial^{top}T^+$ is a stable MOTS. We will distinguish two cases. If $\partial^{top}T^+$ is strictly stable, there exists a foliation $\{\Gamma_s\}_{s\in(-\epsilon,0]}$ of a one sided tubular neighbourhood $\mathcal{W}$ of $\partial^{top}T^+$ in T^+ such that $\Gamma_0 = \partial^{top}T^+$ and all the surfaces $\{\Gamma_s\}_{s<0}$ have $\theta_s^+ < 0$. To see this, simply choose a variation vector $\vec{\nu}$ such that $\vec{\nu}\big|_{\partial^{top}T^+} = \psi\vec{m}$ where ψ is a positive principal eigenfunction of the stability operator $L_{\vec{m}}$ and $\vec{m}$ is the outer direction normal to $\partial^{top}T^+$. Using $\delta_{\vec{\nu}}\theta^+ = L_{\vec{m}}\psi = \lambda\psi > 0$ it follows that the surfaces $\Gamma_s \equiv \varphi_s(\partial^{top}T^+)$ generated by $\vec{\nu}$ are outer trapped for $s \in (-\epsilon, 0)$. Next, define the mapping $\Phi : \mathcal{S}_0 \cap (\mathcal{W} \setminus \partial^{top}T^+) \to (-\epsilon, 0) \subset \mathbb{R}$ which assigns to each point $\mathfrak{p} \in \mathcal{S}_0 \cap (\mathcal{W} \setminus \partial^{top}T^+)$ the corresponding value of the parameter of the foliation $s \in (-\epsilon, 0)$ on $\mathfrak{p}$. Sard Lemma (Lemma B.8) implies that the set of regular values of the mapping Φ is dense in $(-\epsilon, 0) \subset \mathbb{R}$. Select a regular value s_0 as close to 0 as desired. Then, the surface $\tilde{S} \equiv \Gamma_{s_0}$ intersects transversally $\mathcal{S}_0$, as required.

If $\partial^{top}T^+$ is stable but *not strictly stable*, a foliation Γ_s consisting on weakly outer trapped surfaces may not exist. Nevertheless, following [4], a suitable modification of the interior of $\partial^{top}T^+$ in Σ solves this problem. It is important to remark that, in this case, the contradiction which proves the theorem is obtained by applying the Kriele and Hayward Lemma in the modified initial data set. The modification is performed as follows. Consider the same foliation Γ_s as defined above and replace the second fundamental form K on the hypersurface Σ by the following.

$$\tilde{K} = K - \frac{1}{2}\phi(s)\gamma_s, \tag{4.4.1}$$

where $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a $C^{1,1}$ function such that $\phi(s) = 0$ for $s \geq 0$ (so that the data remains unchanged outside $\partial^{top} T^+$) and γ_s is the projector to Γ_s. Then, the outer null expansion of Γ_s computed in the modified initial data set $(\Sigma, g, \tilde{K})$

$$\tilde{\theta}^+[\Gamma_s] = \theta^+[\Gamma_s] - \phi(s),$$

where $\theta^+[\Gamma_s]$ is the outer null expansion of Γ_s in (Σ, g, K). Since $\partial^{top} T^+$ was a stable but not strictly stable MOTS in (Σ, g, K), $\theta^+[\Gamma_s]$ vanishes at least to second order at $s = 0$. On $s \leq 0$, define $\phi(s) = bs^2$ with b a sufficient large constant. It follows that for some $\epsilon > 0$ we have $\tilde{\theta}^+[\Gamma_s] < 0$ on all Γ_s for $s \in (-\epsilon, 0)$. Working with this foliation, Sard Lemma asserts that a weakly outer trapped surface Γ_{s_0} lying as close to $\partial^{top} T^+$ as desired and intersecting $\mathcal{S}_0$ transversally can be chosen in $(\Sigma, g, \tilde{K})$.

Furthermore, the surface $\mathcal{S}_0$ also has non-positive outer null expansion in the modified initial data, at least for s sufficiently close to zero. Indeed, this outer null expansion $\tilde{\theta}^+[\mathcal{S}_0]$ reads $\tilde{\theta}^+[\mathcal{S}_0] = p[\mathcal{S}_0] + \mathrm{tr}_{\mathcal{S}_0} \tilde{K}$. By (4.4.1), we have $\mathrm{tr}_{\mathcal{S}_0} \tilde{K}\big|_{\mathfrak{r}} = \mathrm{tr}_{\mathcal{S}_0} K\big|_{\mathfrak{r}} - \frac{1}{2}\phi(s_{\mathfrak{r}})\mathrm{tr}_{\mathcal{S}_0}\gamma_{s_{\mathfrak{r}}}$, at any point $\mathfrak{r} \in \mathcal{S}_0$, where $s_{\mathfrak{r}}$ is the value of the leaf Γ_s containing $\mathfrak{r}$, i.e. $\mathfrak{r} \in \Gamma_{s_{\mathfrak{r}}}$. Since $\mathrm{tr}_{\mathcal{S}_0}\gamma_s \geq 0$ (because the pull-back of γ_s is positive semi-definite) we have $\mathrm{tr}_{\mathcal{S}_0}\tilde{K} = \mathrm{tr}_{\mathcal{S}_0} K$ for $s \geq 0$ and $\mathrm{tr}_{\mathcal{S}_0}\tilde{K} \leq \mathrm{tr}_{\mathcal{S}_0} K$ for $s < 0$ (small enough). In any case $\tilde{\theta}^+(\mathcal{S}_0) \leq \theta^+(\mathcal{S}_0) = 0$ and we can apply the Kriele and Hayward Lemma to Γ_{s_0} and $\mathcal{S}_0$ to construct a weakly outer trapped surface which is bounding with respect to S_b, lies in the topological closure of the exterior of $\partial^{top} T^+$ and penetrates this exterior somewhere. Since the geometry outside $\partial^{top} T^+$ has not been modified, this gives a contradiction. ■

Remark 1. This theorem has been formulated for outer trapped surfaces instead of weakly outer trapped surfaces. The reason is that in the proof we have used a foliation in the *inside* part of a tubular neighbourhood of $\partial^{top} T^+$. If S satisfies $\theta^+ = 0$, it is possible that $S = \partial\Sigma = \partial^{top} T^+$ and then we would not have room to use this foliation. It follows that the hypothesis of the theorem can be relaxed to $\theta^+ \leq 0$ if one of the following conditions hold:

1. S is not the outermost MOTS.

2. $S \cap \partial\Sigma = \emptyset$.

3. The KID $(\Sigma, g, K; N, \vec{Y}, \tau)$ can be isometrically embedded into another KID $(\hat{\Sigma}, \hat{g}, \hat{K}, \hat{N}, \hat{\vec{Y}}, \hat{\tau})$ with $\partial\Sigma \subset \mathrm{int}(\hat{\Sigma})$

In this case, Theorem 4.4.1 includes Miao's theorem in the particular case of asymptotically flat time-symmetric vacuum static KID with minimal compact boundary. This is because in the time-symmetric case all points with $\lambda = 0$ are fixed points and hence there are no arc-connected components of $\partial^{top}\{\lambda > 0\}$ with $I_1 = 0$ and $Y^i \nabla^{\Sigma}_i \lambda$ is identically zero on $\partial^{top}\{\lambda > 0\}^{ext}$. □

Remark 2. In geometric terms, hypotheses 1 and 2 of the theorem exclude a priori the possibility that $\partial^{top}\{\lambda > 0\}^{ext}$ intersects the white hole Killing horizon at non-fixed points. A similar theorem exists for initial data sets which do not intersect the black hole Killing horizon (more precisely, such that both inequalities in 1 and 2 are satisfied with the reversed inequality signs). The conclusion of the theorem in this case is that no bounding *past* outer trapped surface can intersect $\{\lambda > 0\}^{\text{ext}}$ provided S_b is a *past* outer untrapped barrier (the proof of this statement can be obtained by applying Theorem 4.4.1 to the static KID $(\Sigma, g, -K; -N, \vec{Y}; \rho, -\vec{J}, \tau))$.

No version of this theorem, however, covers the case when $\partial^{top}\{\lambda > 0\}^{ext}$ intersects both the black hole and the white hole Killing horizon. The reason is that, in this setting, $\partial^{top}\{\lambda > 0\}^{ext}$ is, in general, not smooth and we cannot apply the Andersson-Metzger theorem to $\tilde{\Sigma}$. In the next chapter we will address this case in more detail. □

For the particular case of KID possessing an asymptotically flat end we have the following corollary, which is an immediate consequence of Theorem 4.4.1.

Corollary 4.4.2 *Consider a static KID $(\Sigma, g, K; N, \vec{Y}, \tau)$ with a selected asymptotically flat end Σ_0^∞ and satisfying the NEC. Denote by $\{\lambda > 0\}^{ext}$ the connected component of $\{\lambda > 0\}$ which contains the asymptotically flat end Σ_0^∞. Assume that every arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ with $I_1 = 0$ is closed and*

1. *$NY^i \nabla^{\Sigma}_i \lambda \geq 0$ in each arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ containing at least one fixed point.*

2. *$NY^i m_i \geq 0$ in each arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ which contains no fixed points, where $\vec{m}$ is the unit normal pointing towards $\{\lambda > 0\}^{ext}$.*

Then, any bounding (see Definition 2.3.6) outer trapped surface S in Σ cannot intersect $\{\lambda > 0\}^{ext}$.

Chapter 5

Uniqueness of static spacetimes with weakly outer trapped surfaces

5.1 Introduction

In this chapter we will extend the classic static black hole uniqueness theorems to asymptotically flat static KID containing weakly outer trapped surfaces. As emphasized in the previous chapter, the first step for this extension was given by Miao for the particular case of asymptotically flat, time-symmetric, static and vacuum KID, with compact minimal boundary (Theorem 4.1.2). Indeed, our aim of extending the classic uniqueness theorems for static black holes to the quasi-local setting can be reformulated as generalizing Theorem 4.1.2 to non-vanishing matter (as long as the NEC is satisfied) and arbitrary slices (not necessarily time-symmetric) containing weakly outer trapped surfaces. In the previous chapter we obtained a generalization of this result as a confinement result. In this chapter we address the extension of Miao's theorem as a uniqueness result.

As we already know, the most powerful method to prove uniqueness of static black holes is the *doubling method* of Bunting and Masood-ul-Alam. This method was described in some detail in Section 2.4 where we gave a sketch of the proof of the uniqueness theorem for static electro-vacuum black holes. In the present chapter, our strategy will be precisely to recover the framework of the doubling method from an arbitrary static KID containing a weakly outer trapped surface. As it was discussed in Section 2.4, this framework consists of an asymptotically flat spacelike hypersurface Σ with topological boundary $\partial^{top}\Sigma$ which is a closed (i.e. compact and without boundary) embedded topological manifold and such that the static Killing field is causal on Σ and null only on $\partial^{top}\Sigma$. As we pointed

out in Section 2.4, the existence of this topological manifold $\partial^{top}\Sigma$ is ensured precisely by the presence of a black hole. Note that $\partial^{top}\Sigma$ is not required to be smooth.

Hence, our strategy to conclude uniqueness departing from a static KID $(\Sigma, g, K; N, \vec{Y}, \tau)$ with an asymptotically flat end Σ_0^∞ which contains a bounding MOTS S will be therefore to prove that the topological boundary $\partial^{top}\{\lambda > 0\}^{ext}$, where $\{\lambda > 0\}^{ext}$ is the connected component of $\{\lambda > 0\}$ in Σ which contains Σ_0^∞, is a closed embedded topological submanifold. Since a priori MOTS have nothing to do with black holes, $\partial^{top}\{\lambda > 0\}^{ext}$ may fail to be closed (see Figure 5.1) as required in the doubling method. Consequently, throughout this chapter we will study under which conditions we can guarantee that $\partial^{top}\{\lambda > 0\}^{ext}$ is closed. In fact, it turns out that the confinement Theorem 4.4.1 and its Corollary 4.4.2 are already sufficient to conclude that $\partial^{top}\{\lambda > 0\}^{ext}$ is a closed surface. This leads to our first uniqueness result.

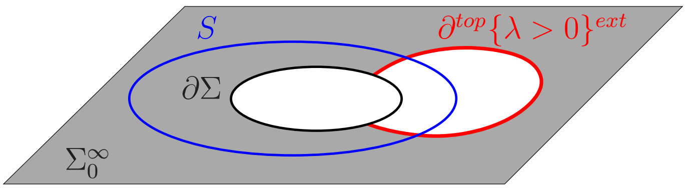


Figure 5.1: The figure illustrates a situation where $\partial^{top}\{\lambda > 0\}^{ext}$ (in red) has non-empty manifold boundary (which lies in $\partial\Sigma$) and, therefore, is not closed. Here, S (in blue) represents a bounding MOTS and the grey region corresponds to $\{\lambda > 0\}^{ext}$. In a situation like this the doubling method cannot be applied.

Theorem 5.1.1 *Consider a static KID $(\Sigma, g, K; N, \vec{Y}, \tau)$ with a selected asymptotically flat end Σ_0^∞ and satisfying the NEC. Assume that Σ possesses an outer trapped surface S which is bounding. Denote by $\{\lambda > 0\}^{ext}$ the connected component of $\{\lambda > 0\}$ which contains the asymptotically flat end Σ_0^∞. If*

1. *Every arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ with $I_1 = 0$ is topologically closed.*

2. *$NY^i\nabla_i^\Sigma\lambda \geq 0$ in each arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ containing at least one fixed point.*

3. *$NY^i m_i \geq 0$ in each arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ which contains no fixed points, where $\vec{m}$ is the unit normal pointing towards $\{\lambda > 0\}^{ext}$.*

4. *The matter model is such that Bunting and Masood-ul-Alam doubling method gives uniqueness of black holes.*

Then, $(\{\lambda > 0\}^{ext}, g, K)$ *is a slice of such a unique spacetime.*

Proof. Proposition 4.3.15 implies that $\partial^{top}\{\lambda > 0\}^{ext}$ is a smooth submanifold with $\theta^{+} = 0$ with respect to the normal pointing towards $\{\lambda > 0\}^{ext}$. We only need to show that $\partial^{top}\{\lambda > 0\}^{ext}$ is closed (i.e. embedded, compact and without boundary) in order to apply hypothesis 4 and conclude uniqueness. By definition of bounding in the asymptotically flat setting (see Definition 2.3.6) we have a compact manifold $\tilde{\Sigma}$ with boundary $\partial\tilde{\Sigma} = S \cup S_b$, where $S_b = \{r = r_0\}$ is a sufficiently large coordinate sphere in Σ_0^∞. Take this sphere large enough so that $\{r \geq r_0\} \subset \{\lambda > 0\}^{ext}$. We are in a setting where all the hypothesis of Theorem 4.4.1 hold. In the proof of this theorem we have shown that $\partial^{top}\{\lambda > 0\}^{ext}$ is embedded and compact. Moreover, $\partial^{top}T^{+}$ lies in the interior $\mathrm{int}(\tilde{\Sigma})$ and does not intersect $\{\lambda > 0\}^{ext}$. This, clearly prevents $\partial^{top}\{\lambda > 0\}^{ext}$ from reaching S, which in turn implies that $\partial^{top}\{\lambda > 0\}^{ext}$ has no boundary. ■

Remark. This theorem applies in particular to static KID which are asymptotically flat, without boundary and have at least two asymptotic ends, as long as conditions 1 to 4 are fulfilled. To see this, recall that an asymptotically flat initial data is the union of a compact set and a finite number of asymptotically flat ends. Select one of these ends Σ_0^∞ and define S to be the union of coordinate spheres with sufficiently large radius on all the other asymptotic ends. This surface is an outer trapped surface which is bounding with respect to Σ_0^∞ and we recover the hypotheses of Theorem 5.1.1. □

Theorem 5.1.1 has been formulated for outer trapped surfaces instead of weakly outer trapped surfaces for the same reason as in Theorem 4.4.1. Consequently, the hypotheses of this theorem can also be relaxed to $\theta^{+} \leq 0$ if one of the following conditions hold: S is not the outermost MOTS, $S \cap \partial\Sigma = \emptyset$, or the KID can be extended. Under these circumstances, this result already extends Miao's theorem as a uniqueness result.

Nevertheless, the theorem above requires several conditions on the boundary $\partial^{top}\{\lambda > 0\}^{ext}$. Since $\partial^{top}\{\lambda > 0\}^{ext}$ is a fundamental object in the doubling procedure, it is rather unsatisfactory to require conditions directly on this object. Out main aim in this chapter is to obtain a uniqueness result which does not involve any a priori restriction on $\partial^{top}\{\lambda > 0\}^{ext}$. As discussed in the previous

chapter, $\partial^{top}\{\lambda > 0\}^{ext}$ is in general not a smooth submanifold (see e.g. Figure 4.1) and the techniques of the previous chapter cannot be applied to conclude that $\partial^{top}\{\lambda > 0\}^{ext}$ is a closed embedded topological submanifold. The key difficulty lies in proving that $\partial^{top}\{\lambda > 0\}^{ext}$ is a manifold without boundary. In the previous theorem, we used the non-penetration property of $\partial^{top}T^+$ into $\{\lambda > 0\}^{ext}$ in order to conclude that $\partial^{top}\{\lambda > 0\}^{ext}$ must lie in the exterior of the bounding outer trapped surface S (which implies that $\partial^{top}\{\lambda > 0\}^{ext}$ is a manifold without boundary). In turn, this non-penetration property was strongly based on the smoothness of $\partial^{top}\{\lambda > 0\}^{ext}$, which we do not have in general. The main problem is therefore: How can we exclude the possibility that $\partial^{top}\{\lambda > 0\}^{ext}$ reaches S in the general case? (see Figure 5.1).

To address this issue we need to understand better the structure of $\partial^{top}\{\lambda > 0\}^{ext}$ (and, more generally, of $\partial^{top}\{\lambda > 0\}$) when conditions 2 and 3 are not satisfied. As we will discuss later, this will force us to view KID as hypersurfaces embedded in a spacetime, instead as abstract objects on their own, as we have done in the previous chapter.

To finish this introduction, let us give a briefly summary of the chapter. In Section 5.2 we define the concept of an *embedded static KID* and present some known results on the structure of the spacetime in the neighbourhood of the fixed points of the isometry. In Section 5.3 we will revisit the study of the properties of $\partial^{top}\{\lambda > 0\}$, this time for embedded static KID. Finally, Section 5.4 is devoted to state and prove the uniqueness theorem for asymptotically flat static spacetimes containing a bounding weakly outer trapped surface.

The results presented in this chapter have been summarized in [31] and will also be sent to publication [30].

5.2 Embedded static KID

We begin this section with the definition of an embedded static KID. Recall that, according to our definitions, a spacetime has **no boundary**.

Definition 5.2.1 *An* **embedded static KID** $(\Sigma, g, K; N, \vec{Y}, \tau)$ *is a static KID, possibly with boundary, which is embedded in a spacetime* $(M, g^{(4)})$ *with static Killing field* $\vec{\xi}$ *such that* $\vec{\xi}|_{\Sigma} = N\vec{n} + \vec{Y}$*, where* $\vec{n}$ *is the unit future directed normal of* Σ *in* M*.*

Remark. If a static KID has no boundary and belongs to a matter model for which the Cauchy problem is well-posed (e.g. vacuum, electro-vacuum, scalar

field, Yang-Mills field, σ-model, etc), it is clear that there exists a spacetime which contains the initial data set as a spacelike hypersurface. Whether this Cauchy development admits or not a Killing vector $\vec{\xi}$ compatible with the Killing data has only been answered in the affirmative for some special matter models, which include vacuum and electro-vacuum [46]. Even in these circumstances, it is at present not known whether the spacetime thus constructed is in fact *static* (i.e. such that the Killing vector $\vec{\xi}$ is integrable). This property is obvious near points where $N \neq 0$ (i.e. points where $\vec{\xi}$ is transverse to Σ), but it is much less clear near fixed points, specially those with $I_1 < 0$. Indeed, these points belong to a totally geodesic closed spacelike surface in the Cauchy development of the initial data set. The points lying in the chronological future of this surface cannot be reached by integral curves of the Killing vector starting on Σ. Proving that the Killing vector is integrable on those points is an interesting and, apparently, not so trivial task. In this thesis we do not explore this problem further and simply work with the definition of embedded static KID stated above. $\square$

In what follows, we will review some useful results concerning the structure of the spacetime near fixed points of the static Killing $\vec{\xi}$.

Proposition 5.2.2 *Let* $(\Sigma, g, K; N, \vec{Y}, \tau)$ *be a static embedded KID and let* $(M, g^{(4)})$ *be the static spacetime where the KID is embedded. Consider a fixed point* $\mathfrak{p} \in \partial^{top}\{\lambda > 0\} \subset \Sigma$ *and let* S_0 *be the connected spacelike surface of fixed points in* M *containing* $\mathfrak{p}$ *(which exists by Theorem 2.4.9). Then, there exists a neighbourhood* $\mathcal{V}$ *of* $\mathfrak{p}$ *in* M *and coordinates* $\{u, v, x^A\}$ *on* $\mathcal{V}$ *such that* $\{x^A\}$ *are coordinates for* $S_0 \cap \mathcal{V}$ *and the spacetime metric takes the Rácz-Wald-Walker form*

$$g^{(4)}_{RWW} = 2G du dv + \gamma_{AB} dx^A dx^B, \tag{5.2.1}$$

where $S_0 \cap \mathcal{V} = \{u = v = 0\}$, ∂_v *is future directed and* G *and* γ_{AB} *are both positive definite and depend smoothly on* $\{w \equiv uv, x^A\}$.

Proof. Theorem 2.4.9 establishes that $\mathfrak{p}$ belongs to a connected, spacelike, smooth surface S_0 which lies in the closure of a non-degenerate Killing horizon. Thus, we can use the Rácz-Wald-Walker construction, see [98], which shows that there exists a neighbourhood $\mathcal{V}$ of $\mathfrak{p}$ and coordinates $\{u, v, x^A\}$ adapted to $S_0 \cap \mathcal{V}$ such that the metric $g^{(4)}$ takes the form

$$g^{(4)} = 2G du dv + 2v H_A dx^A du + \gamma_{AB} dx^A dx^B, \tag{5.2.2}$$

where G, H_A and γ_{AB} depend smoothly on $\{w, x^A\}$. In these coordinates, the Killing vector $\vec{\xi}$ reads

$$\vec{\xi} = c^2 \left(v\partial_v - u\partial_u \right), \tag{5.2.3}$$

where c is a (non-zero) constant and ∂_v is future directed. We only need to prove that staticity implies that $\{u, v, x^A\}$ can be chosen in such a way that $H_A = 0$. A straightforward computation shows that the integrability condition $\boldsymbol{\xi} \wedge d\boldsymbol{\xi} = 0$ is equivalent to the following equations

$$G\partial_w H_A - H_A \partial_w G = 0, \tag{5.2.4}$$

$$H_{[A}\partial_{B]}G + G\partial_{[A}H_{B]} = 0, \tag{5.2.5}$$

$$H_{[A}\partial_w H_{B]} = 0. \tag{5.2.6}$$

Equation (5.2.4) implies $H_A = f_A G$, where f_A depend on x^C. Inserting this in (5.2.5), we get $\partial_{[A} f_{B]} = 0$, which implies (after restricting $\mathcal{V}$ if necessary) the existence of a function $\zeta(x^C)$ such that $f_A = \partial_A \zeta$. Equation (5.2.6) is then identically satisfied. Therefore, staticity is equivalent to

$$H_A(w, x^C) = G(w, x^C)\partial_A \zeta(x^C). \tag{5.2.7}$$

We look for a coordinate change $\{u, v, x^C\} \to \{u', v', x'^C\}$ which preserves the form of the metric (5.2.2) and such that $H'_A = 0$. It is immediate to check that an invertible change of the form

$$\left\{ u = u(u'), v = v(v', x'^C), x^A = x'^A \right\}$$

preserves the form of the metric and transforms H_A as

$$v' H'_A = \frac{du}{du'} \left(\frac{\partial v}{\partial x'^A} G + v H_A \right), \tag{5.2.8}$$

So, we need to impose $G\partial_A v + vH_A = 0$, which in view of (5.2.7), reduces to $\partial_A v + v\partial_A \zeta = 0$. Since $v = v' e^{-\zeta}$ (with v' independent of x^A) solves this equation, we conclude that the coordinate change

$$\left\{ u = u', v = v' e^{-\zeta(x'^C)}, x^A = x'^A \right\}$$

brings the metric into the form (5.2.2) (after dropping the primes). ■

Now, let us consider an embedded static KID in a static spacetime with Rácz-Wald-Walker metric $(\mathcal{V}, g^{(4)}_{RWW})$. Since the vector ∂_v is null on $\mathcal{V}$, it is transverse to $\Sigma \cap \mathcal{V}$ and, therefore, the embedding of $\Sigma \cap \mathcal{V}$ can be written locally as

$$\Sigma : (u, x^A) \to (u, v = \phi(u, x^A), x^A), \tag{5.2.9}$$

where ϕ is a smooth function. A simple computation using (5.2.3) leads to

$$\lambda|_{\Sigma\cap\mathcal{V}} = 2c^A\hat{G}u\phi, \tag{5.2.10}$$

$$N|_{\Sigma\cap\mathcal{V}} = (\phi + u\partial_u\phi)\sqrt{\frac{c^A\hat{G}}{2\partial_u\phi - \hat{G}\partial_A\phi\partial^A\phi}}, \tag{5.2.11}$$

$$\mathbf{Y}|_{\Sigma\cap\mathcal{V}} = c^2\hat{G}\left(\phi du - ud\phi\right). \tag{5.2.12}$$

where $\hat{G} \equiv G(w = u\phi, x^A)$ and indices $A, B, \ldots$ are raised with the inverse of $\hat{\gamma}_{AB} \equiv \gamma_{AB}(w = u\phi, x^A)$.

Since Σ is spacelike, the quantity $2\partial_u\phi - \hat{G}\partial_A\phi\partial^A\phi$ is positive. In particular, this implies that

$$\partial_u\phi > 0, \tag{5.2.13}$$

which will be used later. For the sets $\{u = 0\}$ and $\{\phi = 0\}$ in $\Sigma \cap \mathcal{V}$ we have the following result.

Lemma 5.2.3 *Consider an embedded static KID $(\Sigma, g, K; N, \vec{Y}, \tau)$ and use Rácz-Wald-Walker coordinates $\{u, v, x^A\}$ in a spacetime neighbourhood $\mathcal{V}$ of a fixed point $\mathfrak{p} \in \partial^{top}\{\lambda > 0\} \subset \Sigma$ such that the embedding of Σ reads (5.2.9). Then the sets $\{u = 0\}$ and $\{\phi = 0\}$ in $\Sigma \cap \mathcal{V}$ are both smooth surfaces (not necessarily closed). Moreover, a point $\mathfrak{p} \in \partial^{top}\{\lambda > 0\}$ in $\Sigma \cap \mathcal{V}$ is a non-fixed point if and only if $u\phi = 0$ with either u or ϕ non-zero.*

Proof: The lemma follows directly from the fact that both sets $\{u = 0\}$ and $\{\phi = 0\}$ in Σ are the intersections between Σ and the null smooth embedded hypersurfaces $\{u = 0\}$ and $\{v = 0\}$ in $(\mathcal{V}, \overset{(4)}{g}_{RWW})$, respectively. The second statement of the lemma is a direct consequence of equations (5.2.3) and (5.2.10).

■

5.3 Properties of $\partial^{top}\{\lambda > 0\}$ on an embedded static KID

In this section we will explore in more detail the properties of the set $\partial^{top}\{\lambda > 0\}$ in Σ. In particular, we will study the structure $\partial^{top}\{\lambda > 0\}$ in an embedded KID when no additional hypothesis are made. First, we will briefly recall some results of the previous chapter which will be used below. In Proposition 4.3.10 we showed that an open set of fixed points in $\partial^{top}\{\lambda > 0\}$ in a static KID $(\Sigma, g, K; N, \vec{Y}, \tau)$

is a smooth and totally geodesic surface. Moreover, Lemma 4.3.7 and Proposition 4.3.15 imply that every arc-connected component of the open set of non-fixed points in $\partial^{top}\{\lambda > 0\} \subset \Sigma$ is a smooth submanifold (not necessarily embedded) of Σ and has either $\theta^+ = 0$ or $\theta^- = 0$. The structure of those arc-connected components of $\partial^{top}\{\lambda > 0\}$ having exclusively fixed points or exclusively non-fixed points is therefore clear with no need of additional assumptions. However, for the case of arc-connected components having both types of points an additional assumption on the sign of $NY^i\nabla^{\Sigma}_i\lambda$ was required to conclude smoothness (see Propositions 4.3.14 and 4.3.15). This hypothesis was imposed in order to avoid the existence of *transverse* fixed points in $\partial^{top}\{\lambda > 0\}$ (see stage 1 on the proof of Proposition 4.3.14). Actually, the existence of transverse points is, by itself, not very problematic. Indeed, as we showed in Lemma 4.3.13, the structure of $\partial^{top}\{\lambda > 0\}$ on a neighbourhood of transverse fixed points is well understood and consists of two intersecting branches. The problematic situation happens when a sequence of transverse fixed points tends to a non-transverse point $\mathfrak{p}$. In this case the intersecting branches can have a very complicated limiting behavior at $\mathfrak{p}$. If we consider the non-transverse limit point $\mathfrak{p}$, then we know from the previous chapter (see stage 2 on the proof of Proposition 4.3.14) that locally near $\mathfrak{p}$ there exists coordinates such that $\lambda = Q_0^2 x^2 - \zeta(z^A)$, with ζ a non-negative smooth function. In order to understand the behavior of $\partial^{top}\{\lambda > 0\}$ we need to take the square root of ζ. Under the assumptions of Proposition 4.3.14 we could show that the *positive* square root is C^1. For general non-transverse points, this positive square root is not C^1. In fact, is not clear at all whether there exists any C^1 square root (even allowing this square root to change sign). The following example shows a function ζ which admits no C^1 square root. It is plausible that the equations that are satisfied in a static KID forbid the existence of ζ functions with no C^1 square root. This is, however, a difficult issue and we have not been able to resolve it. This is the reason why we need to restrict ourselves to embedded static KID in this chapter. Assuming the existence of a static spacetime where the KID is embedded, it follows that, irrespectively of the structure of fixed points in Σ, a suitable square root of ζ always exists.

Example. Non-negative functions do not have in general a C^1 square root. A simple example is given by the function $\rho = y^2 + z^2$ on $\mathbb{R}^2$. We know, however, that this type of example cannot occur for the function ζ because the Hessian of ζ must vanish at least on one point where ζ vanishes (and this is obviously not true for ρ).

The following is an example of a non-negative function ζ for which the function

and its Hessian vanish at one point and which admits no C^1 square root. Consider the function $\zeta(y,z) = z^2y^2 + z^4 + f(y)$, where $f(y)$ is a smooth function such that $f(y) = 0$ for $y \geq 0$ and $f(y) > 0$ for $y < 0$. Recall that the set of fixed points consists of the zeros of ζ, and a fixed point is non-transverse if and only if the Hessian of ζ vanishes (see the proof of Proposition 4.3.14). It follows that the fixed points occur on the semi-line $\sigma \equiv \{y \geq 0, z = 0\}$, with $(0,0)$ being non-transverse and $(y > 0, z = 0)$ transverse. Consider the points $\mathfrak{p} = (1,-1)$ and $\mathfrak{q} = (1,1)$. First of all take a curve γ joining them in such a way that it does not intersect σ. It is clear that ζ remains positive along γ and, therefore, its square root cannot change sign (if it is to be continuous). Now consider the curve $\gamma' = \{y = 1, -1 \leq z \leq 1\}$ joining $\mathfrak{p}$ and $\mathfrak{q}$ (which does intersect σ). Since $\zeta\big|_{\gamma'} = z^2(1+z^2)$, the only way to find a C^1 square root is by taking $u = z\sqrt{1+z^2}$, which changes sign from $\mathfrak{p}$ to $\mathfrak{q}$. This is a contradiction to the property above. So, we conclude that no C^1 square root of ζ exists.

Let us see that, in the spacetime setting, this behavior cannot occur. Our first result of this section shows that the set $\partial^{top}\{\lambda > 0\}$ in an embedded KID is a union of compact, smooth surfaces which has one of the two null expansions equal to zero.

Proposition 5.3.1 *Consider an embedded static KID $(\tilde{\Sigma}, g, K; N, \vec{Y}, \tau)$, compact and possibly with boundary $\partial\tilde{\Sigma}$. Assume that every arc-connected component of $\partial^{top}\{\lambda > 0\}$ with $I_1 = 0$ is topologically closed. Then*

$$\partial^{top}\{\lambda > 0\} = \underset{a}{\cup} S_a, \tag{5.3.1}$$

where each S_a is a smooth, compact, connected and orientable surface such that its boundary, if non-empty, satisfies $\partial S_a \subset \partial\tilde{\Sigma}$. Moreover, at least one of the two null expansions of S_a vanishes everywhere.

Proof. Let $\{\mathfrak{S}_\alpha\}$ be the collection of arc-connected components of $\partial^{top}\{\lambda > 0\}$. We know that the quantity I_1 is constant on each $\mathfrak{S}_\alpha$ (see Lemma 4.3.11). Consider an arc-connected component $\mathfrak{S}_d$ of $\partial^{top}\{\lambda > 0\}$ with $I_1 = 0$. Since all points in this component are non-fixed, it follows that $\mathfrak{S}_d$ is a smooth submanifold. Using the hypothesis that arc-connected components with $I_1 = 0$ are topologically closed it follows that $\mathfrak{S}_d$ is, in fact, embedded. Choose $\vec{m}$ to be the unit normal satisfying

$$\vec{Y} = N\vec{m}, \tag{5.3.2}$$

on $\mathfrak{S}_d$. This normal is smooth (because neither $\vec{Y}$ nor N vanish anywhere on $\mathfrak{S}_d$), which implies that $\mathfrak{S}_d$ is orientable. Inserting $\vec{Y} = N\vec{m}$ into equation (4.2.2) and taking the trace it follows

$$p + q = 0. \tag{5.3.3}$$

Consider now a $\mathfrak{S}_\alpha$ with $I_1 \neq 0$. At non-fixed points we know that $\mathfrak{S}_\alpha$ is a smooth embedded surface with $\nabla_i^\Sigma \lambda \neq 0$. On those points, define a unit normal $\vec{m}$ by the condition

$$N\vec{m}(\lambda) > 0 \tag{5.3.4}$$

We also know that $\nabla_i^\Sigma \lambda = 2\kappa Y_i$ where $I_1 = -2\kappa^2$. Let us see that $\mathfrak{S}_\alpha = \mathfrak{S}_{1,\alpha} \cup \mathfrak{S}_{2,\alpha}$, where each $\mathfrak{S}_{1,\alpha}$ and $\mathfrak{S}_{2,\alpha}$ is a smooth, embedded, connected and orientable surface. To that aim, define

$$\begin{aligned} \mathfrak{S}_{1,\alpha} &= \{\mathfrak{p} \in \mathfrak{S}_\alpha \text{ such that } \kappa\big|_{\mathfrak{p}} > 0\} \cup \{\text{ fixed points in } \mathfrak{S}_\alpha\}, \\ \mathfrak{S}_{2,\alpha} &= \{\mathfrak{p} \in \mathfrak{S}_\alpha \text{ such that } \kappa\big|_{\mathfrak{p}} < 0\} \cup \{\text{ fixed points in } \mathfrak{S}_\alpha\}. \end{aligned}$$

Notice that the fixed points are assigned to *both* sets. It is clear that at non-fixed points, both $\mathfrak{S}_{1,\alpha}$ and $\mathfrak{S}_{2,\alpha}$ are smooth embedded surfaces. Let $\mathfrak{q}$ be a fixed point in $\mathfrak{S}_\alpha$ and consider the Rácz-Wald-Walker coordinate system discussed in Proposition 5.2.2. The points in $\mathfrak{S}_\alpha \cap \mathcal{V}$ are characterized by $\{u\phi = 0\}$ (due to (5.2.10)). Inserting (5.2.10) and (5.2.12) into $\nabla_i^\Sigma \lambda = 2\kappa Y_i$ yields, at any non-fixed point $\mathfrak{q}' \in \mathfrak{S}_\alpha \cap \mathcal{V}$,

$$2c^2 \left(\phi du + u d\phi\right)|_{\mathfrak{q}'} = 2\kappa \left(\phi du - u d\phi\right)|_{\mathfrak{q}'}.$$

Since $du \neq 0$ (because u is a coordinate) and $d\phi \neq 0$ (see equation (5.2.13)) we have

$$\begin{aligned} \kappa > 0 \quad &\text{on} \quad \{u = 0, \phi \neq 0\}, \\ \kappa < 0 \quad &\text{on} \quad \{u \neq 0, \phi = 0\}. \end{aligned} \tag{5.3.5}$$

Consequently, the non-fixed points in $\mathfrak{S}_{1,\alpha} \cap \mathcal{V}$ are defined by the condition $\{u = 0, \phi \neq 0\}$ and the non-fixed points in $\mathfrak{S}_{2,\alpha} \cap \mathcal{V}$ are defined by the condition $\{u \neq 0, \phi = 0\}$. It is then clear that $\mathfrak{S}_{1,\alpha} \cap \mathcal{V} = \{u = 0\}$ and $\mathfrak{S}_{2,\alpha} \cap \mathcal{V} = \{\phi = 0\}$, which are smooth embedded surfaces. It remains to see that the unit normal $\vec{m}$, which has been defined only at non-fixed points via (5.3.4), extends to a well-defined normal to all of $\mathfrak{S}_{1,\alpha}$ and $\mathfrak{S}_{2,\alpha}$ (see Figure 5.2).

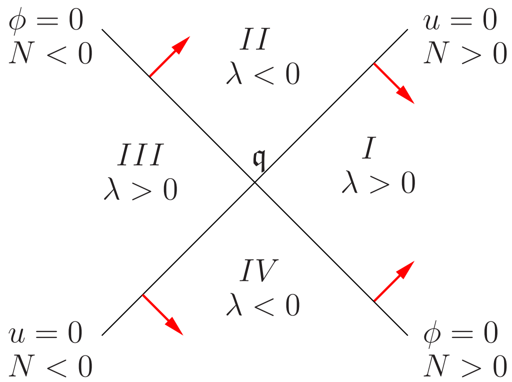


Figure 5.2: In the Rácz-Wald-Walker coordinate system we define four open regions by $I = \{u > 0\} \cap \{\phi > 0\}, II = \{u < 0\} \cap \{\phi > 0\}, III = \{u < 0\} \cap \{\phi < 0\}, IV = \{u > 0\} \cap \{\phi < 0\}$. The normal on its boundaries which satisfies (5.3.4) is depicted in red color. It is clear graphically that these normals extend smoothly to the fixed points on the hypersurfaces $\{u = 0\}$ and $\{\phi = 0\}$, such as $\mathfrak{q}$ in the figure. This figure is, however, only schematic because one dimension has been suppressed and fixed points need not be isolated in general. A formal proof that $\vec{m}$ extends smoothly in all cases is given in the text.

This requires to check that the condition (5.3.4), when evaluated on $\mathcal{V}$ defines a normal which extends smoothly to the fixed points. Consider first the points $\{u \neq 0, \phi = 0\}$. The unit normal to this surface is $\vec{m} = \epsilon |\nabla^\Sigma \phi|_g^{-1} \nabla^\Sigma \phi$ where $\epsilon = \pm 1$ and may, a priori, depend on the point. Since

$$
\begin{aligned}
N|_{\{u\neq 0, \phi=0\}} &= u\partial_u \phi \sqrt{\frac{c^4 \hat{G}}{2\partial_u \phi - \hat{G}\partial_A \phi \partial^A \phi}}, \\
\nabla_i^\Sigma \lambda|_{\{u\neq 0, \phi=0\}} &= 2c^4 \hat{G} u \nabla_i^\Sigma \phi,
\end{aligned}
$$

expression (5.3.4) implies

$$
0 < N\vec{m}(\lambda)|_{\{u\neq 0, \phi=0\}} = 2\epsilon c^4 \hat{G} u^2 \partial_u \phi |\nabla^\Sigma \phi|_g \sqrt{\frac{c^4 \hat{G}}{2\partial_u \phi - \hat{G}\partial_A \phi \partial^A \phi}}.
$$

Hence $\epsilon = 1$ at all points on $\{u \neq 0, \phi = 0\}$. Thus the normal vector reads $\vec{m} = |\nabla^\Sigma \phi|_g^{-1} \nabla^\Sigma \phi$ at non-fixed points, and this field clearly extends smoothly to all points on $\mathfrak{S}_{1,\alpha} \cap \mathcal{V}$. This implies, in particular, that $\mathfrak{S}_{1,\alpha}$ is orientable.

The argument for $\mathfrak{S}_{2,\alpha}$ is similar. Consider now the points $\{u = 0, \phi \neq 0\}$. The unit vector normal to this surface is $\vec{m} = \epsilon' |\nabla^\Sigma u|_g^{-1} \nabla^\Sigma u$ where $\epsilon' = \pm 1$.

Using (5.2.10) and (5.2.11) in (5.3.4) gives now

$$0 < N\vec{m}(\lambda)|_{\{u=0,\phi\neq 0\}} = 2\epsilon' c^4 \hat{G}\phi^2 |\nabla^\Sigma u|_g \sqrt{\frac{c^4\hat{G}}{2\partial_u\phi - \hat{G}\partial_A\phi\partial^A\phi}},$$

which implies $\epsilon' = 1$ all points on $\{u = 0, \phi \neq 0\}$. The normal vector is $\vec{m} = |\nabla^\Sigma u|_g^{-1}\nabla^\Sigma u$ which again extends smoothly to all points on $\mathfrak{S}_{2,\alpha} \cap \mathcal{V}$. As before, $\mathfrak{S}_{2,\alpha}$ is orientable.

Let us next check that $\mathfrak{S}_{1,\alpha}$ has $\theta^+ = 0$ and $\mathfrak{S}_{2,\alpha}$ has $\theta^- = 0$ (both with respect to the normal $\vec{m}$ defined above). On open sets of fixed points this is a trivial consequence of Proposition 4.3.10 which implies both $p = q = 0$. To discuss the non-fixed points, we need an expression for $\vec{Y}$ in terms of $\vec{m}$. Let $\vec{Y} = \epsilon'' N\vec{m}$, where $\epsilon'' = \pm 1$. Using $\vec{Y} = \frac{1}{2\kappa}\nabla^\Sigma\lambda$, we have

$$\frac{\epsilon''}{2\kappa}|\nabla^\Sigma\lambda|_g^2 = \epsilon''\vec{Y}(\lambda) = N\vec{m}(\lambda) > 0$$

Hence $\epsilon'' = \text{sign}(\kappa)$ and

$$\vec{Y} = \text{sign}(\kappa)N\vec{m}. \tag{5.3.6}$$

Inserting this into (4.2.2) and taking the trace, it follows

$$\text{sign}(\kappa)p + q = 0 \tag{5.3.7}$$

This implies that $\theta^+ = p + q = 0$ at non-fixed points of $\mathfrak{S}_{1,\alpha}$ and $\theta^- = -p + q = 0$ at non-fixed points at $\mathfrak{S}_{2,\alpha}$. At fixed points not lying on open sets, equations $\theta^+ = 0$ (resp. $\theta^- = 0$) follow by continuity once we know that $\mathfrak{S}_{1,\alpha}$ (resp. $\mathfrak{S}_{2,\alpha}$) is smooth with a smooth unit normal.

The final step is to prove that $\mathfrak{S}_{1,\alpha}$ and $\mathfrak{S}_{2,\alpha}$ are topologically closed. Let us first show that $\mathfrak{S}_\alpha$ is topologically closed. Consider a sequence of points $\{\mathfrak{p}_i\}$ in $\mathfrak{S}_\alpha$ converging to $\mathfrak{p}$. It is clear that $\mathfrak{p} \in \partial^{top}\{\lambda > 0\}$, so we only need to check that we have not moved to another arc-connected component. If $\mathfrak{p}$ is a non-fixed point, then $\{\lambda = 0\}$ is a defining function for $\partial^{top}\{\lambda > 0\}$ near $\mathfrak{p}$ and the statement is obvious. If $\mathfrak{p}$ is a fixed point, we only need to use the Rácz-Wald-Walker coordinate system near $\mathfrak{p}$ to conclude that no change of arc-connected component can occur in the limit. To show that each $\mathfrak{S}_{1,\alpha}$, $\mathfrak{S}_{2,\alpha}$ is topologically closed, assume now that $\mathfrak{p}_i$ is a sequence on $\mathfrak{S}_{1,\alpha}$. If the limit $\mathfrak{p}$ is a fixed point, it belongs to $\mathfrak{S}_{1,\alpha}$ by definition. If the limit $\mathfrak{p}$ is a non-fixed point, we can take a subsequence $\{\mathfrak{p}_i\}$ of non-fixed points. Since κ remains constant on the sequence, it takes the same value in the limit, which shows that $\mathfrak{p} \in \mathfrak{S}_{1,\alpha}$, i.e. $\mathfrak{S}_{1,\alpha}$ is topologically closed.

The surfaces S_a in the statement of the theorem are the collection of $\{\mathfrak{S}_d\}$ having $I_1 = 0$ and the collection of pairs $\{\mathfrak{S}_{1,\alpha},\ \mathfrak{S}_{2,\alpha}\}$ for the arc-connected components $\mathfrak{S}_\alpha$ with $I_1 \neq 0$. The statement that $\partial S_a \subset \partial\tilde{\Sigma}$ is obvious. ■

Remark 1. In this proof we have tried to avoid using the existence of a spacetime where $(\Sigma, g, K; N, \vec{Y}, \tau)$ is embedded as much as possible. The only essential information that we have used from the spacetime is that, near fixed points, λ can be written as the product of two smooth functions with non-zero gradient, namely u and ϕ. This is the square root of ζ that we mentioned above. To see this, simply note that if a square root h of ζ exists, then $\lambda = Q_0 x^2 - \zeta = Q_0^2 x - h^2 = (Q_0 x - h)\,(Q_0 x + h))$. The functions $Q_0 x \pm h$ have non-zero gradient and are, essentially, the functions u and ϕ appearing the Rácz-Wald-Walker coordinate system. □

Remark 2. The assumption of every arc-connected component of $\partial^{top}\{\lambda > 0\}$ with $I_1 = 0$ being topologically closed is needed to ensure that these arc-connected components are embedded and compact. From a spacetime perspective, this hypothesis avoids the existence of non-embedded degenerate Killing prehorizons which would imply that, on an embedded KID, the arc-connected components of $\partial^{top}\{\lambda > 0\}$ which intersect these prehorizons could be non-embedded or non-compact (see Figure 2.7 in Chapter 2). Although it has not been proven, it may well be that non-embedded Killing prehorizons cannot exist. A proof of this fact would allow us to drop automatically this hypothesis in the theorem. □

We are now in a situation where we can prove that $\partial^{top}\{\lambda > 0\}^{ext} = \partial^{top}T^+$ under suitable conditions on the trapped region and on the topology of $\tilde{\Sigma}$. This result is the crucial ingredient for our uniqueness result later. The strategy of the proof is, once again, to assume that $\partial^{top}\{\lambda > 0\}^{ext} \neq \partial^{top}T^+$ and to construct a bounding weakly outer trapped surface outside $\partial^{top}T^+$. This time, the surface we use to perform the smoothing is more complicated than $\partial^{top}\{\lambda > 0\}^{ext}$, which we used in the previous chapter. The newly constructed surface will have vanishing outer null expansion and will be closed and oriented. However, we cannot guarantee a priori that it is bounding. To address this issue we impose a topological condition on $\mathrm{int}(\tilde{\Sigma})$ which forces that all closed and orientable surfaces separate the manifold into disconnected subsets. This topological condition involves the first homology group $H_1(\mathrm{int}(\tilde{\Sigma}), \mathbb{Z}_2)$ with coefficients in $\mathbb{Z}_2$ and imposes that this homology group is trivial. More precisely, the theorem that we will invoke is due

to Feighn [56] and reads as follows

Theorem 5.3.2 (Feighn, 1985) *Let $\mathcal{N}$ and $\mathcal{M}$ be manifolds without boundary of dimension n and $n+1$ respectively. Let $f : \mathcal{N} \to \mathcal{M}$ be a proper immersion (an immersion is proper if inverse images of compact sets are compact). If $H_1(\mathcal{M}, \mathbb{Z}_2) = 0$ then $\mathcal{M} \setminus f(\mathcal{N})$ is not connected. Moreover, if two points $\mathfrak{p}_1$ and $\mathfrak{p}_2$ can be joined by an embedded curve intersecting $f(\mathcal{N})$ transversally at just one point, then $\mathfrak{p}_1$ and $\mathfrak{p}_2$ belong to different connected components of $\mathcal{M} \setminus f(\mathcal{N})$.*

The proof of this theorem requires that all embedded closed curves in $\mathcal{M}$ are the boundary of an embedded compact surface. This is a consequence of $H_1(\mathcal{M}, \mathbb{Z}_2) = 0$ and this is the only place where this topological condition enters into the proof. This allows us to understand better what topological restriction we are really imposing on $\mathcal{M}$, namely that every closed embedded curve is the boundary of a compact surface.

Without entering into details of algebraic topology, we just notice that $H_1(\mathcal{M}, \mathbb{Z}_2)$ vanishes if $H_1(\mathcal{M}, \mathbb{Z}) = 0$ (see e.g. Theorem 4.6 in [115]) and, in turn, this is automatically satisfied in simply connected manifolds (see e.g. Theorem 4.29 in [101]).

Theorem 5.3.3 *Consider an embedded static KID $(\tilde{\Sigma}, g, K; N, \vec{Y}, \tau)$ compact, with boundary $\partial\tilde{\Sigma}$ and satisfying the NEC. Suppose that the boundary can be split into two non-empty disjoint components $\partial\tilde{\Sigma} = \partial^-\tilde{\Sigma} \cup \partial^+\tilde{\Sigma}$ (neither of which are necessarily connected). Take $\partial^+\tilde{\Sigma}$ as a barrier with interior $\tilde{\Sigma}$ and assume $\theta^+[\partial^-\tilde{\Sigma}] \leq 0$ and $\theta^+[\partial^+\tilde{\Sigma}] > 0$ Let T^+, T^- be, respectively, the weakly outer trapped and the past weakly outer trapped regions of $\tilde{\Sigma}$. Assume also the following hypotheses:*

1. *Every arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ with $I_1 = 0$ is topologically closed.*
2. *$\lambda|_{\partial^+\tilde{\Sigma}} > 0$.*
3. *$H_1\left(int(\tilde{\Sigma}), \mathbb{Z}_2\right) = 0$.*
4. *T^- is non-empty and $T^- \subset T^+$.*

Denote by $\{\lambda > 0\}^{ext}$ the connected component of $\{\lambda > 0\}$ which contains $\partial^+\tilde{\Sigma}$. Then

$$\partial^{top}\{\lambda > 0\}^{ext} = \partial^{top}T^+,$$

Therefore, $\partial^{top}\{\lambda > 0\}^{ext}$ is a non-empty stable MOTS which is bounding with respect to $\partial^+\tilde{\Sigma}$ and, moreover, it is the outermost bounding MOTS.

Proof. After replacing $\vec{\xi} \to -\vec{\xi}$ if necessary, we can assume without loss of generality that $N > 0$ on $\{\lambda > 0\}^{ext}$. From Theorem 2.2.31, we know that the boundary of the weakly outer trapped region T^+ in $\tilde{\Sigma}$ (which is non-empty because $\theta^+[\partial^- \tilde{\Sigma}] \leq 0$) is a stable MOTS which is bounding with respect to $\partial^+ \tilde{\Sigma}$. $\partial^{top} T^-$ is also non-empty by assumption.

Since we are dealing with embedded KID, and all spacetimes are boundaryless in this thesis, it follows that $(\Sigma, g, K; N, \vec{Y}, \tau)$ can be extended as a smooth hypersurface in $(M, g^{(4)})$[1]. Working on this extended KID allows us to assume without loss of generality that $\partial^{top} T^+$ and $\partial^{top} T^-$ lie in the *interior* of $\tilde{\Sigma}$. This will be used when invoking the Kriele and Hayward smoothing procedure below.

First of all, Theorem 3.4.10 implies that $\partial^{top}\{\lambda > 0\}^{ext}$ cannot lie completely in T^+ and intersect the topological interior $\overset{\circ}{T}{}^+$ (here is where we use the NEC). Therefore, either $\partial^{top}\{\lambda > 0\}^{ext}$ intersects the exterior of $\partial^{top} T^+$ or they both coincide. We only need to exclude the first possibility. Suppose, that $\partial^{top}\{\lambda > 0\}^{ext}$ penetrates into the exterior of $\partial^{top} T^+$. Let $\{\mathfrak{U}\}$ be the collection of arc-connected components of $\partial^{top}\{\lambda > 0\}$ which have a non-empty intersection with $\partial^{top}\{\lambda > 0\}^{ext}$. In Proposition 5.3.1 we have shown that $\{\mathfrak{U}\}$ decomposes into a union of smooth surfaces S_a. Define its unit normal $\vec{m}'$ as the smooth normal which points into $\{\lambda > 0\}^{ext}$ at points on $\partial^{top}\{\lambda > 0\}^{ext}$. This normal exists because all S_a are orientable. By (5.3.4) and the fact that $N > 0$ on $\{\lambda > 0\}^{ext}$, we have that on the surfaces S_a with $I_1 \neq 0$, the normal $\vec{m}'$ coincides with the normal $\vec{m}$ defined in the proof of Proposition 5.3.1. On the surfaces S_a with $I_1 = 0$, this normal coincides with $\vec{m}$ provided $\vec{Y}$ points into $\{\lambda > 0\}^{ext}$, see (5.3.2). Since, by assumption, $\partial^{top}\{\lambda > 0\}^{ext}$ penetrates into the exterior of T^+, it follows that there is at least one S_a with penetrates into the exterior of T^+. Let $\{S_{a'}\}$ be the subcollection of $\{S_a\}$ consisting on the surfaces which penetrate into the exterior of $\partial^{top} T^+$. A priori, none of the surfaces $S_{a'}$ need to satisfy $p + q = 0$ with respect to the normal $\vec{m}'$. However, one of the following two possibilities must occur:

1. There exists at least one surface, say S_0, in $\{S_{a'}\}$ containing a point $\mathfrak{q} \in \partial^{top}\{\lambda > 0\}^{ext}$ such that $\vec{Y}|_{\mathfrak{q}}$ points inside $\{\lambda > 0\}^{ext}$, or

2. All surfaces in $\{S_{a'}\}$ have the property that, for any $\mathfrak{q} \in S_{a'} \cap \partial^{top}\{\lambda > 0\}^{ext}$ we have $\vec{Y}|_{\mathfrak{q}}$ is either zero, or it points outside $\{\lambda > 0\}^{ext}$.

[1]Simply consider $\partial\tilde{\Sigma}$ as a surface in $(M, g^{(4)})$ and let $\vec{m}$ the be the spacetime normal to $\partial\tilde{\Sigma}$ which is tangent to $\tilde{\Sigma}$. Take a smooth hypersurface containing $\partial\tilde{\Sigma}$ and tangent to $\vec{m}$. This hypersurface extends $(\Sigma, g, K; N, \vec{Y}, \tau)$. It is clear that the extension can be selected as smooth as desired.

In case 1, we have that S_0 satisfies $p + q = 0$ with respect to the normal $\vec{m}'$. Indeed, we either have that S_0 satisfies $I_1 = 0$ or $I_1 \neq 0$. If $I_1 = 0$ then, since $\vec{Y}$ points into $\{\lambda > 0\}^{ext}$, we have that $\vec{m}$ and $\vec{m}'$ coincide. Since S_0 satisfies $p + q = 0$ with respect to $\vec{m}$ (see (5.3.3)) the statement follows. If $I_1 \neq 0$ then $\kappa > 0$ on S_0 (from (5.3.6) and the fact that $\vec{m} = \vec{m}'$). Thus, $p + q = 0$ follows from (5.3.7).

In case 2, all surfaces $\{S_{a'}\}$ satisfy $\theta^- = -p + q = 0$ with respect to $\vec{m}'$ and we cannot find a MOTS outside $\partial^{top}T^+$. However, under assumption 3, we have $T^- \subset T^+$ and hence each $S_{a'}$ penetrates into the exterior of T^-. We can therefore reduce case 2 to case 1 by changing the time orientation (or simply replacing θ^+ and T^+ by θ^- and T^- in the argument below).

Let us therefore restrict ourselves to case 1. We know that S_0 either has no boundary, or the boundary is contained in $\partial^-\tilde{\Sigma}$. If S_0 has no boundary, simply rename this surface to S_1. When S_0 has a non-empty boundary, it is clear that S_0 must intersect $\partial^{top}T^+$. We can then use the smoothing procedure by Kriele and Hayward (see Lemma 3.5.1) to construct a closed surface S_1 penetrating into the exterior of $\partial^{top}T^+$ and satisfying $\theta^+ \leq 0$ with respect to a normal $\vec{m}''$ which coincides with $\vec{m}'$ outside the region where the smoothing is performed (see Figure 5.3). As discussed in the previous chapter, when S_0 and $\partial^{top}T^+$ do not intersect transversally we need to apply the Sard Lemma to surfaces inside $\partial^{top}T^+$. If $\partial^{top}T^+$ is only marginally stable, a suitable modification of the initial data set inside $\partial^{top}T^+$ is needed. The argument was discussed in depth at the end of the proof of Theorem 4.4.1 and applies here without modification.

So, in either case (i.e. irrespectively of whether S_0 has boundary of not), we have a closed surface S_1 penetrating into the exterior of $\partial^{top}T^+$ and satisfying $\theta^+ \leq 0$ with respect to $\vec{m}''$. Here we apply the topological hypothesis 3 ($H_1(\mathrm{int}(\tilde{\Sigma}), \mathbb{Z}_2) = 0$). Indeed S_1 is a closed manifold embedded into $\mathrm{int}(\tilde{\Sigma})$. Since S_1 is compact, its embedding is obviously proper. Thus, the theorem by Feighn [56] (Theorem 5.3.2) implies that $\mathrm{int}(\tilde{\Sigma}) \setminus S_1$ has at least two connected components. It is clear that one of the connected components Ω of $\mathrm{int}(\tilde{\Sigma}) \setminus S_1$ contains $\partial^+\tilde{\Sigma}$. Moreover, by Feighn's theorem there is a tubular neighbourhood of S_1 which intersects this connected component only to one side of S_1. Consequently, $\overline{\Omega}$ is a compact manifold with boundary $\partial\overline{\Omega} = S_1 \cap \partial^+\Sigma$. If follows that S_1 is bounding with respect to $\partial^+\tilde{\Sigma}$. The choice of $\vec{m}''$ is such that $\vec{m}''$ points towards $\partial^+\tilde{\Sigma}$. Consequently S_1 is a weakly outer trapped surface which is bounding with respect to $\partial^+\tilde{\Sigma}$ penetrating into the exterior of $\partial^{top}T^+$, which is impossible. ■

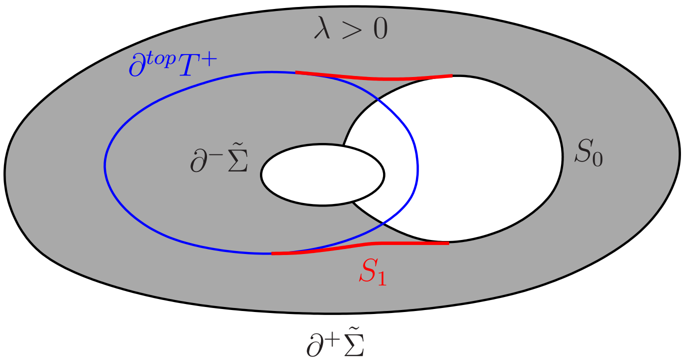


Figure 5.3: The figure illustrates the situation when S_0 has boundary. The grey region represents the region with $\lambda > 0$ in $\tilde{\Sigma}$. In this case we use the smoothing procedure of Kriele and Hayward to construct a smooth surface S_1 from S_0 and $\partial^{top}T^+$ (in blue). The red lines represent precisely the part of S_1 which comes from smoothing S_0 and $\partial^{top}T^+$.

Remark 1. If the hypothesis $T^- \subset T^+$ is not assumed, then the possibility 2 in the proof of the Theorem would not lead to a contradiction (at least with our method of proof). To understand this better, without the assumption $T^- \subset T^+$ it may happen a priori that all the surfaces $S_{a'}$ (which have $\theta^- = 0$ and penetrate in the exterior of $\partial^{top}T^+$) are fully contained in T^-. A situation like this is illustrated in Figure 5.4, where $\partial^{top}T^-$ intersects $\partial^{top}T^+$. It would be interesting to either prove this theorem without the assumption $T^- \subset T^+$ or else find a counterexample of the statement $\partial^{top}\{\lambda > 0\}^{ext} = \partial^{top}T^+$ when assumption 4 is dropped. The problem, however, appears to be difficult. $\square$

5.4 The uniqueness result

Finally, we are ready to state and prove the uniqueness result for static spacetimes containing trapped surfaces.

Theorem 5.4.1 *Let $(\Sigma, g, K; N, \vec{Y}, \tau)$ be an embedded static KID with a selected asymptotically flat end Σ_0^∞ and satisfying the NEC. Assume that Σ possesses a weakly outer trapped surface S which is bounding. Assume the following:*

1. *Every arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ with $I_1 = 0$ is topologically closed.*

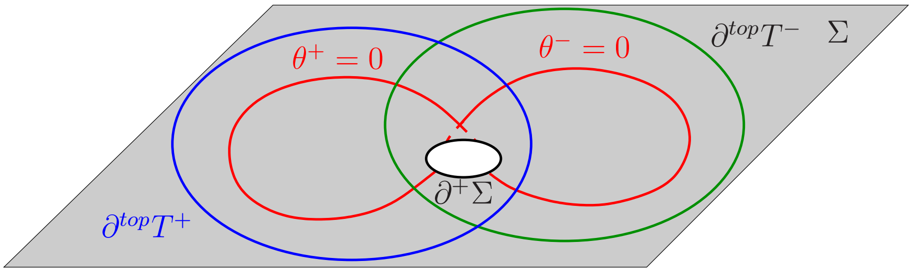


Figure 5.4: The figure illustrates a hypothetical situation where $T^+ \subset T^-$ does not hold and the conclusions of the Theorem 5.3.3 would not be true. The red continuous line represents the set $\partial^{top}\{\lambda > 0\}^{ext}$ which is composed by a smooth surface with $\theta^+ = 0$, lying inside of $\partial^{top}T^+$ (in blue) and partly outside of $\partial^{top}T^-$ (in green), and a smooth surface with $\theta^- = 0$, which lies partly outside of $\partial^{top}T^+$ and inside of $\partial^{top}T^-$.

2. *T^- is non-empty and $T^- \subset T^+$.*

3. *$H_1(\Sigma, \mathbb{Z}_2) = 0$.*

4. *The matter model is such that Bunting and Masood-ul-Alam doubling method for time-symmetric initial data sets gives uniqueness of black holes.*

Then $(\Sigma \setminus T^+, g, K)$ is a slice of such a unique spacetime.

Proof. Take a coordinate sphere $S_b \equiv \{r = r_0\}$ in the asymptotically flat end Σ_0^∞ with r_0 large enough so that $\lambda > 0$ on $\{r \geq r_0\} \subset \Sigma_0^\infty$ and all the surfaces $\{r = r_1\}$ with $r_1 \geq r_0$ are outer untrapped with respect to the unit normal pointing towards increasing r. S_b is a barrier with interior $\Omega_b = \Sigma \setminus \{r > r_0\}$.

Take $\tilde{\Sigma}$ to be the topological closure of the exterior of S in Ω_b. Then define $\partial^-\tilde{\Sigma} = S$ and $\partial^+\tilde{\Sigma} = S_b$. Let $\{\lambda > 0\}^{ext}$ be the connected component of $\{\lambda > 0\} \subset \tilde{\Sigma}$ containing S_b. All the hypothesis of Theorem 5.3.3 are satisfied and we can conclude $\partial^{top}\{\lambda > 0\}^{ext} = \partial^{top}T^+$. This implies that the manifold $\Sigma \setminus T^+$ is an asymptotically flat spacelike hypersurface with topological boundary $\partial^{top}(\Sigma \setminus T^+)$ which is compact and embedded (moreover, it is smooth) such that the static Killing vector is timelike on $\Sigma \setminus T^+$ and null on $\partial^{top}(\Sigma \setminus T^+)$. Under these assumptions, the doubling method of Bunting and Masood-ul-Alam [23] can be applied. Hence, hypothesis 4 gives uniqueness. ■

Remark 1. In contrast to Theorems 4.4.1 and 5.1.1, this result has been formulated for weakly outer trapped surfaces instead of outer trapped surfaces.

As mentioned in the proof of Theorem 5.3.3 this is because, $(\Sigma, g, K; N, \vec{Y}, \tau)$ being an embedded static KID, it can be extended smoothly as a hypersurface in the spacetime. It is clear however, that we are hiding the possible difficulties in the definition of *embedded static KID*. Consider, for instance, a *static KID* with boundary and assume that the KID is vacuum. The Cauchy problem is of course well-posed for vacuum initial data. However, since Σ has boundary, the spacetime constructed by the Cauchy development also has boundary and we cannot a priori guarantee that the static KID is an embedded static KID (this would require extending the spacetime, which is as difficult – or more – than extending the initial data).

Consequently, Theorem 5.4.1 includes Miao's theorem in vacuum as a particular case only for vacuum static KID for which either (i) S is not the outermost MOTS, (ii) $S \cap \partial\Sigma = \emptyset$ or (iii) the KID can be extended as a vacuum static KID. Despite this subtlety, we emphasize that all the other conditions of the theorem are fulfilled for asymptotically flat, time-symmetric vacuum KID with a compact minimal boundary. Indeed, condition 4 is obviously satisfied for vacuum. Moreover, the property of time-symmetry implies that all points with $\lambda = 0$ are fixed points and hence no arc-connected component of $\partial^{top}\{\lambda > 0\}$ with $I_1 = 0$ exists. Thus, condition 1 is automatically satisfied. Time-symmetry also implies $T^- = T^+$ and condition 2 is trivial. Finally, the region outside the outermost minimal surface in a Riemannian manifold with non-negative Ricci scalar is $\mathbb{R}^3$ minus a finite number of closed balls (see e.g. [70]). This manifold is simply connected and hence satisfies condition 3. □

Remark 2. Condition 4 in the theorem could be replaced by a statement of the form

4'. The matter model is such that static black hole initial data implies uniqueness, where a *black hole static initial data* is an asymptotically flat static KID possibly with boundary with an asymptotically flat end Σ_0^∞ such that $\partial^{top}\{\lambda > 0\}^{ext}$ (defined as the connected component of $\{\lambda > 0\}$ containing the asymptotic region in Σ_0^∞) is a topological manifold without boundary and compact.

The Bunting and Masood-ul-Alam method is, at present, the most powerful method to prove uniqueness under the circumstances of 4'. However, if a new method is invented, Theorem 5.4.1 would still give uniqueness. □

Remark 3. A comment on the condition $T^- \subset T^+$ is in order. First of all, in the static regime, T^+ and T^- are expected to be the intersections of both the black and the white hole with $\tilde{\Sigma}$. Therefore, the hypothesis $T^- \subset T^+$ could be understood as the requirement that the first intersection, as coming from $\partial^+\tilde{\Sigma}$, of $\tilde{\Sigma}$ with an event horizon occurs with the black hole event horizon. Therefore, this hypothesis is similar to the hypotheses on $\partial^{top}\{\lambda > 0\}^{ext}$ made in Theorem 4.3.15. However, there is a fundamental difference between them: The hypothesis $T^- \subset T^+$ is an hypothesis on the weakly outer trapped regions which, a priori, have nothing to do with the location and properties of $\partial^{top}\{\lambda > 0\}^{ext}$. In a physical sense, the existence of past weakly outer trapped surfaces in the spacetime reveals the presence of a white hole region. Moreover, given a (3+1) decomposition of a spacetime satisfying the NEC, the Raychaudhuri equation implies that T^- shrinks to the future while T^+ grows to the future (see [1]) ("grow" and "shrink" is with respect to any timelike congruence in the spacetime). It is plausible that by letting the initial data evolve sufficiently long, only the black hole event horizon is intersected by Σ. The uniqueness theorem 5.4.1 could be applied to this evolved initial data. Although this requires much less global assumptions than for the theorem that ensures that no MOTS can penetrate into the domain of outer communications, it still requires some control on the evolution of the initial data. In any case, we believe that the condition $T^- \subset T^+$ is probably not necessary for the validity of the theorem. It is an interesting open problem to analyze this issue further. $\square$

We conclude with a trivial corollary of Theorem 5.4.1, which is nevertheless interesting.

Corollary 5.4.2 *Let $(\Sigma, g, K = 0; N, \vec{Y} = 0; \rho, \vec{J} = 0, \tau_{ij}; \vec{E})$ be a time-symmetric electrovacuum embedded static KID, i.e a static KID with an electric field $\vec{E}$ satisfying*

$$\nabla^\Sigma_i E^i = 0, \quad \rho = |\vec{E}|^2_g, \quad \tau_{ij} = |\vec{E}|^2 g_{ij} - 2E_iE_j.$$

Let $\Sigma = \mathcal{K} \cup \Sigma_0^\infty$ where $\mathcal{K}$ is a compact and Σ_0^∞ is an asymptotically flat end and assume that $\partial\Sigma \neq \emptyset$ with mean curvature with respect to the normal which points inside Σ satisfying $p \leq 0$. Then $(\Sigma \backslash T^+, g, K = 0; N, \vec{Y} = 0, \rho, \vec{J} = 0, \tau_{ij}, \vec{E})$ can be isometrically embedded in the Reissner-Nordström spacetime with $M > |Q|$, where M is the ADM mass of (Σ, g) and Q is the total electric charge of $\vec{E}$, defined as $Q = \frac{1}{4\pi}\int_{S_{r_0}} E^i m_i \eta_{S_{r_0}}$ where $S_{r_0} \subset \Sigma_0^\infty$ is the coordinate sphere $\{r = r_0\}$ and $\vec{m}$ its unit normal pointing towards infinity.

Remark. The standard Majumdar-Papapetrou spacetime cannot occur because it possesses degenerate Killing horizons which are excluded in the hypotheses of the corollary (recall that, by Proposition 2.4.11, degenerate Killing horizons implies cylindrical ends in time-symmetric slices). □

Chapter 6

A counterexample of a recent proposal on the Penrose inequality

6.1 Introduction

In this chapter we will give a counter-example of the Penrose inequality proposed by Bray and Khuri in [20].

As discussed in Chapter 2, in a consistent attempt [20] to prove the standard Penrose inequality (equation (2.3.6)) in the general case (i.e. non-time-symmetric), Bray and Khuri were led to conjecture a new version of the Penrose inequality in terms of the outermost generalized apparent horizon (see Definition 2.2.17) as follows.

$$M_{ADM} \geq \sqrt{\frac{|S_{out}|}{16\pi}}, \tag{6.1.1}$$

where M_{ADM} is the ADM mass of a spacelike hypersurface Σ, which contains an asymptotically flat end Σ_0^∞, and $|S_{out}|$ denotes the area of the outermost bounding generalized apparent horizon S_{out} in Σ. As we already remarked in Section 2.3, this inequality has several convenient properties such as the invariancy under time reversals, no need of taking the minimal area enclosure of S_{out}, and the facts that it is stronger than (2.3.6) and covers a larger number of slices of Kruskal with equality than (2.3.6). Furthermore, it also has good analytical properties which potentially can lead to its proof in the general case. Indeed, Bray and Khuri proved that if a certain system of PDE admits solutions with the right boundary behavior, then (6.1.1) follows.

Nevertheless, as we also pointed out in Section 2.3, inequality (6.1.1) is not directly supported by cosmic censorship. In fact, it is not difficult to obtain particular situations where S_{out} lies, at least partially, outside the event horizon, as

for example for a slice Σ in the Kruskal spacetime for which $\partial^{top}T^+$ and $\partial^{top}T^-$ intersect transversally. In this case, Eichmair's theorem (Theorem 2.2.32) implies that there exists a $C^{2,\alpha}$ outermost generalized apparent horizon lying, at least partially, in the domain of outer communications of the Kruskal spacetime.

Thus, it becomes natural to study the outermost generalized apparent horizon in slices of this type in order to check whether (6.1.1) holds or not. Surprisingly, the result we will find is that there are examples for which inequality (6.1.1) turns out to be violated. More precisely,

Theorem 6.1.1 *In the Kruskal spacetime with mass $M_{Kr} > 0$, there exist asymptotically flat, spacelike hypersurfaces with an outermost generalized apparent horizon S_{out} satisfying $|S_{out}| > 16\pi M_{Kr}^2$.*

For the systems of PDE proposed in [20], this means that a general existence theory cannot be expected with boundary conditions compatible with generalized apparent horizons. However, simpler boundary conditions (e.g. compatible with future and past apparent horizons) are not ruled out. This may in fact simplify the analysis of these equations.

The results on this chapter have been published in [28], [29].

6.2 Construction of the counterexample

Let us consider the Kruskal spacetime of mass $M_{Kr} > 0$ with metric

$$ds^2 = \frac{32M_{Kr}^3}{r} e^{-r/2M_{Kr}} d\hat{u}d\hat{v} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right),$$

where $r(\hat{u}\hat{v})$ solves the implicit equation

$$\hat{u}\hat{v} = \frac{r - 2M_{Kr}}{2M_{Kr}} e^{r/2M_{Kr}}. \tag{6.2.1}$$

In this metric $\partial_{\hat{v}}$ is future directed and $\partial_{\hat{u}}$ is past directed. The region $\{\hat{u} > 0, \hat{v} > 0\}$ defines the domain of outer communications and $\{\hat{u} = 0\}$, $\{\hat{v} = 0\}$ define, respectively, the black hole and white hole event horizons. Consider the one-parameter family of axially-symmetric embedded hypersurfaces $\Sigma_\epsilon = \mathbb{R} \times \mathbb{S}^2$, with intrinsic coordinates $\hat{y} \in \mathbb{R}$, $x \in [-1, 1]$, $\phi \in [0, 2\pi]$, defined by the embedding

$$\Sigma_\epsilon \equiv \{\hat{u} = \hat{y} - \epsilon x, \hat{v} = \hat{y} + \epsilon x, \cos\theta = x, \phi = \phi\}.$$

Inserting this embedding functions into equation (6.2.1) we get

$$\hat{y}^2 - \epsilon^2 x^2 = \frac{r - 2M_{Kr}}{2M_{Kr}} e^{r/2M_{Kr}}, \tag{6.2.2}$$

from which it is immediate to show that, for $|\epsilon| < 1$, Σ_ϵ does not touch the Kruskal singularity ($r = 0$) for any value of $\{\hat{y}, x\}$ in their coordinate range. It is also immediate to check that the hypersurfaces Σ_ϵ are smooth everywhere, included the north and south poles defined by $|x| = 1$. It is straightforward to prove that the induced metric g_ϵ on Σ_ϵ is positive definite and satisfies (for ϵ is small enough) $g_\epsilon = dr^2 + r^2 \left(\frac{dx^2}{1-x^2} + (1-x^2)d\phi^2 \right) + O^{(2)}(\frac{1}{r})$, where r is defined in (6.2.2). Consequently, the hypersurfaces Σ_ϵ are spacelike and asymptotically flat. Let us select $\Sigma_{\epsilon 0}^{\infty}$ to be the asymptotically flat end of the region $\{\hat{u} > 0, \hat{v} > 0\}$.

The discrete isometry of the Kruskal spacetime defined by $\{\hat{u}, \hat{v}\} \to \{\hat{v}, \hat{u}\}$ implies that under reflection with respect to the equatorial plane, i.e. $(\hat{y}, x, \phi) \to (\hat{y}, -x, \phi)$, the induced metric of Σ_ϵ remains invariant, while the second fundamental form of Σ_ϵ changes sign. The latter is due to the fact that Σ_ϵ is defined by $\hat{u} - \hat{v} + 2\epsilon x = 0$ and hence the future directed unit normal to Σ_ϵ is proportional (with metric coefficients which only depend on uv and x^2) to $d\hat{u} - d\hat{v} + 2\epsilon dx$ and, therefore, it changes sign under a reflection $(\hat{y}, x, \phi) \to (\hat{y}, -x, \phi)$ and a simultaneous spacetime isometry $\{\hat{u}, \hat{v}\} \to \{\hat{v}, \hat{u}\}$ (notice that this isometry reverses the time orientation). Let us denote by Σ_ϵ^+ the intersection of Σ_ϵ with the domain of outer communications $\{\hat{u} > 0, \hat{v} > 0\}$, which is given by $\{\hat{y} - |\epsilon x| > 0\}$. For $\epsilon \neq 0$, $\partial^{top}\Sigma_\epsilon^+$ is composed by a portion of the black hole event horizon and a portion of the white hole event horizon. Moreover, $\partial^{top}T^+$ is given by $\{\hat{y} - \epsilon x = 0\}$, while $\partial^{top}T^-$ is $\{\hat{y} + \epsilon x = 0\}$ so that these surfaces intersect transversally on the circumference $\{\hat{y} = 0, x = 0\}$ provided $\epsilon \neq 0$[1]. By Eichmair's theorem (Theorem 2.2.32), there exists a $C^{2,\alpha}$ outermost generalized apparent horizon S_{out} which is bounding and contains both $\partial^{top}T^+$ and $\partial^{top}T^-$. Uniqueness implies that this surface must be axially symmetric and have equatorial symmetry. In what follows we will estimate the area of S_{out} from below . To that aim we will proceed in two steps. Firstly, we will prove that an axial and equatorially symmetric generalized apparent horizon $\hat{S}_\epsilon$ of spherical topology and lying in a sufficiently small neighbourhood of $\{\hat{y} = 0\}$ exists (provided ϵ is small enough) and determine its embedding function. In the second step we will compute its area and prove that it is smaller or equal than the area of the outermost generalized apparent horizon S_{out}.

[1]A graphic example of this type of hypersurface was already given in Figure 4.1 (where one spatial dimension was suppressed), where the portions of S intersecting the black hole event horizon and the white hole event horizon represent part of the sets $\partial^{top}T^+$ and $\partial^{top}T^-$, respectively. The tip of S at the intersection with the bifurcation surface S_0 corresponds to the circumference $\{\hat{y} = 0, x = 0\}$.

6.2.1 Existence and embedding function

This subsection is devoted to prove the existence of $\hat{S}_\epsilon$ and to calculate its embedding function up to first order in ϵ. For that, we will consider surfaces S_ϵ of spherical topology defined by embedding functions $\{\hat{y} = y(x,\epsilon), x = x, \phi = \phi\}$ in Σ_ϵ and satisfying $y(-x,\epsilon) = y(x,\epsilon)$. Since the outermost generalized apparent horizon is known to be $C^{2,\alpha}$ it is natural to consider the spaces of functions

$$U^{m,\alpha} \equiv \left\{ y \in C^{m,\alpha}(\mathbb{S}^2) : \partial_\phi y = 0, y(-x) = y(x) \right\},$$

i.e. the spaces of m-times differentiable functions on the unit sphere, with Hölder continuous m-th derivatives with exponent $\alpha \in (0,1]$ and invariant under the axial Killing vector on $\mathbb{S}^2$ and under reflection about the equatorial plane. Each space $U^{m,\alpha}$ is a closed subset of the Banach space $C^{m,\alpha}(\mathbb{S}^2)$ and hence a Banach space itself. Let $I \subset \mathbb{R}$ be the closed interval where ϵ takes values. The expression that defines a generalized apparent horizon is $p - |q| = 0$, where p is the mean curvature of the corresponding surface S_ϵ in Σ_ϵ with respect to the direction pointing into $\Sigma_{\epsilon 0}^\infty$ and q is the trace on S_ϵ of the pull-back of the second fundamental form K of Σ_ϵ. For each function $y \in U^{2,\alpha}$ the expression $p - |q|$ defines a non-linear map $f : U^{2,\alpha} \times I \to U^{0,\alpha}$. Thus, we are looking for solutions $y \in U^{2,\alpha}$ of the equation $f = 0$.

We know that when $\epsilon = 0$, the hypersurface Σ_ϵ is totally geodesic, which implies $q = 0$ for any surface on it. Consequently, all generalized apparent horizons on $\Sigma_{\epsilon=0}$ satisfy $p = 0$ and are, in fact, minimal surfaces. The only closed minimal surface in $\Sigma_{\epsilon=0}$ is the bifurcation surface $S_0 = \{\hat{u} = 0, \hat{v} = 0\}$. Thus, the equation $f(y,\epsilon) = 0$ has $y = 0$ as the unique solution when $\epsilon = 0$. It becomes natural to use the implicit function theorem for Banach spaces to show that there exists a unique solution $y \in U^{2,\alpha}$ of $f = 0$ in a neighbourhood of $y = 0$ for ϵ small enough. To apply the implicit function theorem it will be necessary to know the explicit form of the linearization of the differential equation $f(y,\epsilon) = 0$. The following lemma gives precisely the explicit form of f up to first order in ϵ.

Lemma 6.2.1 *Let Σ_ϵ be the one-parameter family of axially-symmetric hypersurfaces embedded in the Kruskal spacetime with mass $M_{Kr} > 0$, with intrinsic coordinates $\hat{y} \in \mathbb{R}$, $x \in [-1,1]$, $\phi \in [0,2\pi]$, defined by*

$$\Sigma_\epsilon \equiv \left\{ \hat{u} = \hat{y} - \epsilon x, \hat{v} = \hat{y} + \epsilon x, \cos\theta = x, \phi = \phi \right\}.$$

Consider the surfaces $S_\epsilon \subset \Sigma_\epsilon$ defined by $\{\hat{y} = y(x), x, \phi\}$ where the embedding

function has the form $y = \epsilon Y$*, with* $Y \in U^{m,\alpha}(\mathbb{S}^2)$*. Then,* p *and* q *satisfy*

$$p(y=\epsilon Y, \epsilon) = \frac{1}{M_{Kr}\sqrt{e}} L[Y(x)]\epsilon + O(\epsilon^2), \qquad (6.2.3)$$

$$q(y=\epsilon Y, \epsilon) = -\frac{1}{M_{Kr}\sqrt{e}} 3x\epsilon + O(\epsilon^2), \qquad (6.2.4)$$

where $L[z(x)] \equiv -(1-x^2)\ddot{z}(x) + 2x\dot{z}(x) + z(x)$ *and where the dot denotes derivative with respect to* x*.*

Proof. The proof is by direct computation. Let us define $H = \frac{32M_{Kr}^3}{r} e^{-r/2M_{Kr}}$, $Q = r^2$ and $x = \cos\theta$, so that the Kruskal metric takes the form

$$g^{(4)} = H d\hat{u} d\hat{v} + \frac{Q}{1-x^2} dx^2 + (1-x^2) Q d\phi^2.$$

The induced metric g_ϵ on Σ_ϵ is

$$g_\epsilon = \hat{H} d\hat{y}^2 + \left(\frac{\hat{Q}}{1-x^2} - \epsilon^2 \hat{H} \right) dx^2 + (1-x^2)\hat{Q} d\phi^2, \qquad (6.2.5)$$

where $\hat{H}$, $\hat{Q}$ are obtained from H, Q by expressing r in terms of $(\hat{y}, x)$ according to (6.2.2). The induced metric γ_ϵ on S_ϵ satisfies

$$\gamma_\epsilon = \left[\frac{\tilde{Q}}{1-x^2} + \epsilon^2 \left(\dot{Y}^2(x) - 1 \right) \tilde{H} \right] dx^2 + (1-x^2)\tilde{Q} d\phi^2, \qquad (6.2.6)$$

where $\tilde{H}$, $\tilde{Q}$ are obtained from $\hat{H}$ and $\hat{Q}$ by inserting $\hat{y} = \epsilon Y(x)$. Firstly, let us deal with the computation of $p = -m_i \gamma_\epsilon^{AB} \nabla^{\Sigma_\epsilon}_{\vec{e}_A} e^i_B$, where $\mathbf{m}$ is the unit vector tangent to Σ_ϵ normal to S_ϵ which points to the asymptotically flat end in $\{\hat{u} > 0, \hat{v} > 0\}$ and $\{\vec{e}_A\}$ is a basis for TS_ϵ. In our coordinates

$$\vec{e}_x = \partial_x + \epsilon \dot{Y}(x) \partial_{\hat{y}},$$
$$\vec{e}_\phi = \partial_\phi.$$

The unit normal is therefore

$$\mathbf{m} = \sqrt{\frac{\tilde{H}\left(\tilde{Q} - \epsilon^2(1-x^2)\tilde{H}\right)}{\tilde{Q} + \epsilon^2(1-x^2)(\dot{Y}^2 - 1)\tilde{H}}} \left(d\hat{y} - \epsilon \dot{Y}(x) dx \right). \qquad (6.2.7)$$

Since γ_ϵ is diagonal, we only need to calculate $\nabla^{\Sigma_\epsilon}_{\vec{e}_x} e^{\hat{y}}_x$, $\nabla^{\Sigma_\epsilon}_{\vec{e}_\phi} e^{\hat{y}}_\phi$, $\nabla^{\Sigma_\epsilon}_{\vec{e}_x} e^x_x$ and $\nabla^{\Sigma_\epsilon}_{\vec{e}_\phi} e^x_\phi$

up to first order. The results are the following.

$$\nabla^{\Sigma_\epsilon}_{\vec{e}_x} e^{\hat{y}}_x = -\frac{\partial_{\hat{y}}\hat{Q}}{2(1-x^2)\tilde{H}} + \epsilon\left(\ddot{Y} + \dot{Y}\partial_x \ln \hat{H}\right) + O(\epsilon^2), \tag{6.2.8}$$

$$\nabla^{\Sigma_\epsilon}_{\vec{e}_x} e^{x}_x = \frac{2x + (1-x^2)\partial_x \ln \hat{Q}}{2(1-x^2)} + \epsilon \dot{Y}\partial_{\hat{y}} \ln \hat{Q} + O(\epsilon^2), \tag{6.2.9}$$

$$\nabla^{\Sigma_\epsilon}_{\vec{e}_\phi} e^{\hat{y}}_\phi = -\frac{(1-x^2)\partial_{\hat{y}}\hat{Q}}{2\tilde{H}}, \tag{6.2.10}$$

$$\nabla^{\Sigma_\epsilon}_{\vec{e}_\phi} e^{x}_\phi = \frac{(1-x^2)\left(2x - (1-x^2)\partial_x \ln \hat{Q}\right)}{2} + O(\epsilon^2), \tag{6.2.11}$$

where $\partial_{\hat{y}}\hat{Q}$ means taking derivative with respect to $\hat{y}$ of $\hat{Q}$ and afterwards, substituting $\hat{y} = \epsilon Y(x)$ (and similarly for the other derivatives).

In order to compute the derivatives of $\hat{H}$ and $\hat{Q}$, we need to calculate the derivatives $\partial_{\hat{y}} r(\hat{y}, x)$ and $\partial_x r(\hat{y}, x)$. This can be done by taking derivatives of (6.2.2) with respect to x and $\hat{y}$, which gives,

$$\partial_{\hat{y}} r = \epsilon \frac{8M_{Kr}^2}{r} e^{-r/2M_{Kr}} Y,$$
$$\partial_x r = -\epsilon^2 \frac{8M_{Kr}^2}{r} e^{-r/2M_{Kr}} x.$$

At $\epsilon = 0$ we have $y = 0$ and equation (6.2.2) gives $r\big|_{S_{\epsilon=0}} = 2M_{Kr}$. Then $r\big|_{S_\epsilon} = 2M_{Kr} + O(\epsilon)$ This allows us to compute the derivatives of $\hat{H}$ and $\hat{Q}$ up to first order in ϵ. The result is

$$\partial_x \hat{H} = O(\epsilon^2),$$
$$\partial_{\hat{y}} \hat{Q} = \epsilon \frac{16M_{Kr}^2}{e} Y + O(\epsilon^2),$$
$$\partial_x \hat{Q} = O(\epsilon^2).$$

Inserting these equations into (6.2.6), (6.2.7), (6.2.8), (6.2.9), (6.2.10) and (6.2.11), and putting all these results together, we finally obtain that $p = -m_i \gamma_\epsilon^{AB} \nabla^{\Sigma_\epsilon}_{\vec{e}_A} e^i_B$ satisfies (6.2.3).

Next, we will study $q = \gamma_\epsilon^{AB} e^i_A e^j_B K_{ij}$, where K is the second fundamental form of Σ_ϵ with respect to the future directed unit normal. Since, γ_ϵ is diagonal, we just have to compute $e^i_x e^j_x K_{ij} = \dot{y}^2 K_{yy} + 2\dot{y} K_{xy} + K_{xx}$ and $e^i_\phi e^j_\phi K_{ij} = K_{\phi\phi}$ up to first order. To that aim, it is convenient to take coordinates $\{T = \frac{1}{2}(\hat{v} - \hat{u}), \hat{y} = \frac{1}{2}(\hat{v} + \hat{u}), x, \phi\}$ in the Kruskal spacetime for which the metric $g^{(4)}$ is diagonal. In these coordinates Σ_ϵ is defined by $\{T = \epsilon x, \hat{y}, x, \phi\}$ and the future directed unit

normal to Σ_ϵ reads

$$\mathbf{n} = \sqrt{\frac{\hat{H}\hat{Q}}{\hat{Q} - \epsilon^2(1-x^2)\hat{H}}}\,(-dT + \epsilon dx)\,.$$

The computation of the second fundamental form is straightforward and gives

$$\dot{y}^2 K_{yy} = O(\epsilon^2), \tag{6.2.12}$$
$$2\dot{y} K_{xy} = O(\epsilon^2), \tag{6.2.13}$$
$$K_{xx} = \sqrt{\tilde{H}}\left[\frac{\partial_T Q'}{2(1-x^2)\tilde{H}} - \epsilon\frac{x}{1-x^2}\right] + O(\epsilon^2) \tag{6.2.14}$$
$$K_{\phi\phi} = \sqrt{\tilde{H}}\left[\frac{(1-x^2)\partial_T Q'}{2\tilde{H}} - \epsilon(1-x^2)x\right] + O(\epsilon^2), \tag{6.2.15}$$

where we have denoted by Q' the function obtained from Q by expressing r in terms of $(T, \hat{y})$ according to $\hat{u}\hat{v} = \hat{y}^2 - T^2 = \frac{r-2M_{Kr}}{2M_{Kr}} e^{r/2M_{Kr}}$. This expression also allows us to compute $\partial_T Q'$ which, on Σ_ϵ (where $T = \epsilon x$) and using $r = 2M_{Kr} + O(\epsilon)$, takes the form

$$\partial_T Q' = -\epsilon\frac{16M_{Kr}^2}{e}x + O(\epsilon^2).$$

Inserting this into (6.2.14) and (6.2.15), and using (6.2.6), it is a matter of simple computation to show that $q = \gamma_\epsilon^{AB} e^i_A e^j_B K_{ij}$ satisfies (6.2.4). ■

From this lemma we conclude that $f(y = \epsilon Y, \epsilon) \equiv p(y = \epsilon Y, \epsilon) - |q(y = \epsilon Y, \epsilon)|$ reads

$$f(y = \epsilon Y, \epsilon) = \frac{1}{M_{Kr}\sqrt{e}}\,(L[Y(x)] - 3|x|)\epsilon + O(\epsilon^2). \tag{6.2.16}$$

The implicit function theorem requires the operator f to have a continuous Fréchet derivative and the partial derivative $D_y f|_{(y=0,\epsilon=0)}$ to be an isomorphism (see Appendix B). The problem is not trivial in our case because the appearance of $|x|$ makes the Fréchet derivative of f potentially discontinuous[2]. However, the problem can be solved considering a suitable modification of f, as we discuss in detail next.

Proposition 6.2.2 *There exists a neighborhood $\tilde{I} \subset I$ of $\epsilon = 0$ such that $f(y, \epsilon) = 0$ admits a solution $y(x, \epsilon) \in U^{2,\alpha}(\mathbb{S}^2)$ for all $\epsilon \in \tilde{I}$. Moreover, $y(x, \epsilon)$ is C^1 in ϵ and satisfies $y(x, \epsilon = 0) = 0$.*

[2] We thank M. Khuri for pointing out this issue.

Proof. Firstly, let us consider surfaces S_ϵ in Σ_ϵ defined by $\{\hat{y} = y(x,\epsilon), x, \phi\}$ such that the embedding function has the form $y = \epsilon Y$, where $Y \in U^{2,\alpha}$. Since we are considering surfaces with axial symmetry, neither p nor q depend on ϕ. Let η^μ denote the spacetime coordinates, z^i the coordinates on Σ_ϵ, x^A the coordinates on S_ϵ, $\eta^\mu(z^i)$ the embedding functions of Σ in M (which depend smoothly on z^i), and $z^i(x^A)$ the embedding functions of S in Σ (which depend smoothly on x^A). Thus, by definition, we have

$$p(x,\epsilon) = -\gamma^{AB} m_i \left[\frac{\partial^2 z^i}{\partial x^A \partial x^B} + {\Gamma^{\Sigma_\epsilon}}^i_{jk}(z(x)) \frac{\partial z^j}{\partial x^A} \frac{\partial z^k}{\partial x^B} \right],$$

where ${\Gamma^{\Sigma_\epsilon}}^i_{jk}$ are the Christoffel symbols of Σ_ϵ. In this expression all terms depend smoothly on $(\dot{y}(x), y(x), x, \epsilon)$, except $\frac{\partial^2 z^i}{\partial x^A \partial x^B}$ which also depends on $\ddot{y}(x)$. Therefore, p can be viewed as a smooth function of $(\ddot{y}(x), \dot{y}(x), y(x), x, \epsilon)$. Similarly, by definition,

$$q(x,\epsilon) = -\gamma^{AB} n_\mu e^i_A e^j_B \left[\frac{\partial^2 \eta^\mu}{\partial z^i \partial z^j} + \Gamma^\mu_{\nu\beta}(\eta(z)) \frac{\partial \eta^\nu}{\partial z^i} \frac{\partial \eta^\beta}{\partial z^j} \right] \Bigg|_{z^i = z^i(x^A)},$$

where all terms depend smoothly on $(\dot{y}(x), y(x), x, \epsilon)$ Therefore, setting $y = \epsilon Y$ and since both p and q are $O(\epsilon)$ (see equations (6.2.3) and (6.2.4)), we can write

$$p = \epsilon \mathcal{P}(Y(x), \dot{Y}(x), \ddot{Y}(x), x, \epsilon)$$

and

$$q = \epsilon \mathcal{Q}(Y(x), \dot{Y}(x), x, \epsilon),$$

where $\mathcal{P} : \mathbb{R}^3 \times [-1,1] \times I \to \mathbb{R}$ and $\mathcal{Q} : \mathbb{R}^2 \times [-1,1] \times I \to \mathbb{R}$ are smooth functions. Moreover, the function $\mathcal{Q}$ has the symmetry $\mathcal{Q}(x_1, x_2, x_3, x_4) = -\mathcal{Q}(x_1, -x_2, -x_3, x_4)$, which reflects the fact that the extrinsic curvature of Σ_ϵ changes sign under a transformation $x \to -x$ and the symmetry $Y(-x) = Y(x)$. Let us write $P(Y,\epsilon)(x) \equiv \mathcal{P}(Y(x), \dot{Y}(x), \ddot{Y}(x), x, \epsilon)$ and similarly $Q(Y,\epsilon)(x) \equiv \mathcal{Q}(Y(x), \dot{Y}(x), x, \epsilon)$.

Now, instead of f, let us consider the functional $F : U^{2,\alpha} \times I \to U^{0,\alpha}$ defined by $F(Y,\epsilon) = P(Y,\epsilon) - |Q(Y,\epsilon)|$. This functional has the property that, for $\epsilon > 0$, the solutions of $F(Y,\epsilon) = 0$ correspond exactly to the solutions of $f(y,\epsilon) = 0$ via the relation $y = \epsilon Y$. Moreover, the functional F is well-defined for all $\epsilon \in I$, in particular at $\epsilon = 0$. Therefore, by proving that $F = 0$ admits solutions in a neighbourhood of $\epsilon = 0$, we will conclude that $f = 0$ admits solutions for $\epsilon > 0$ and the solutions will in fact belong to a neighbourhood of $y = 0$ since $y = \epsilon Y$.

In order to show that F admits solutions we will use the implicit function theorem. Equation (6.2.16) yields

$$F(Y, \epsilon = 0)(x) = c\,(L[Y(x)] - 3|x|) \tag{6.2.17}$$

where c is the constant $1/(M_{Kr}\sqrt{e})$ and $L[Y] \equiv -(1-x^2)\ddot{Y} + 2x\dot{Y} + Y$. As it is well-known the eigenvalue problem $(1-x^2)\ddot{z}(x) - 2x\dot{z}(x) + \lambda z(x) = 0$ has non-trivial smooth solutions on $[-1, 1]$ (the Legendre polynomials) if and only if $\lambda = l(l+1)$, with $l \in \mathbb{N} \cup \{0\}$. Thus, the kernel of $L[Y]$ (for which $\lambda = 1$) is $Y = 0$. We conclude that L is an isomorphism between $U^{2,\alpha}$ and $U^{0,\alpha}$. Let $Y_1 \in U^{2,\alpha}$ be the unique solution of the equation $L[Y] = 3|x|$. For later use, we note that $Q(Y_1, \epsilon = 0) = -3cx$ (see equation (6.2.4)). This vanishes *only* at $x = 0$. This is the key property that allows us to prove that F is $C^1(U^{2,\alpha} \times I)$.

The $C^1(U^{2,\alpha} \times I)$ property of the functional $P(Y, \epsilon)$ is immediate from Theorem B.3 in the Appendix B. More subtle is to show that $|Q|$ is $C^1(U^{2,\alpha} \times I)$ in a suitable neighbourhood of $(Y_1, \epsilon = 0)$. Let $r_0 > 0$ and define

$$\mathcal{V}_{r_0} = \{(Y, \epsilon) \in U^{2,\alpha} \times I : \|(Y - Y_1, \epsilon)\|_{U^{2,\alpha} \times I} \leq r_0\}. \tag{6.2.18}$$

First of all we need to show that $|Q|$ is Fréchet-differentiable on $\mathcal{V}_{r_0}$, i.e. that for all $(Y, \epsilon) \in \mathcal{V}_{r_0}$ there exists a bounded linear map $D_{Y,\epsilon}|Q| : U^{2,\alpha} \times I \to U^{0,\alpha}$ such that, for all $(Z, \delta) \in U^{2,\alpha} \times I$, $|Q(Y + Z, \epsilon + \delta)| - |Q(Y, \epsilon)| = D_{Y,\epsilon}|Q|(Z, \delta) + R_{Y,\epsilon}(Z, \delta)$ where $\|R_{Y,\epsilon}(Z, \delta)\|_{U^{0,\alpha}} = o(\|(Z, \delta)\|_{U^{2,\alpha} \times I})$. The key observation is that, by choosing r_0 small enough in Definition 6.2.18, we have

$$|Q(Y, \epsilon)(x)| = -\sigma(x) Q(Y, \epsilon)(x) \quad \forall (Y, \epsilon) \in \mathcal{V}_{r_0}, \tag{6.2.19}$$

where $\sigma(x)$ is the *sign* function, (i.e. $\sigma(x) = +1$ for $x \geq 0$ and $\sigma(x) = -1$ for $x < 0$). To show this we need to distinguish two cases: when x lies in a sufficiently small neighbourhood $(-\varepsilon, \varepsilon)$ of 0 and when x lies outside this neighbourhood. Consider first the latter case. As already mentioned, we have $Q(Y_1, \epsilon = 0) = -3cx$ which is negative for $x > 0$ and positive for $x < 0$. Taking r_0 small enough, and using that $\mathcal{Q}$ is a smooth function of its arguments it follows that the inequalities $Q(Y_1, \epsilon) < 0$ for $x \geq \varepsilon$ and $Q(Y_1, \epsilon) > 0$ for $x \leq -\varepsilon$ still hold for any $(Y, \epsilon) \in \mathcal{V}_{r_0}$. For the points $x \in (-\varepsilon, \varepsilon)$, the function $Q(Y, \epsilon)(x)$ is odd in x, so it passes through zero at $x = 0$. Hence, the relation (6.2.19) holds in $(-\varepsilon, \varepsilon)$ provided we can prove that $Q(Y, \epsilon)$ is strictly decreasing at $x = 0$. But this follows immediately from the fact that $\frac{dQ(Y_1, \epsilon = 0)}{dx}\big|_{x=0} = -3c$ and $\mathcal{Q}$ is a smooth function of its arguments.

From its definition, it follows that $Q(Y, \epsilon)(x)$ is $C^{1,\alpha}$ (note that only first derivatives of Y enter in $\mathcal{Q}$) and that the functional $Q(Y, \epsilon)$ has Fréchet derivative

(see Theorem B.3 in Appendix B)

$$D_{Y,\epsilon}Q(Z,\delta)(x) = A_{Y,\epsilon}(x)Z(x) + B_{Y,\epsilon}(x)\dot{Z}(x) + C_{Y,\epsilon}(x)\delta,$$

where $A_{Y,\epsilon}(x) \equiv \partial_1 \mathcal{Q}|_{(Y(x),\dot{Y}(x),x,\epsilon)}$, $B_{Y,\epsilon}(x) \equiv \partial_2 \mathcal{Q}|_{(Y(x),\dot{Y}(x),x,\epsilon)}$ and $C_{Y,\epsilon}(x) \equiv \partial_4 \mathcal{Q}|_{(Y(x),\dot{Y}(x),x,\epsilon)}$. We note that these three functions are $C^{1,\alpha}$ and that $A_{Y,\epsilon}$, $C_{Y,\epsilon}$ are odd, while $B_{Y,\epsilon}$ is even (as a consequence of the symmetries of $\mathcal{Q}$). Defining the linear map

$$D_{Y,\epsilon}|Q|(Z,\delta) \equiv -\sigma(A_{Y,\epsilon}Z + B_{Y,\epsilon}\dot{Z} + C_{Y,\epsilon}\delta),$$

it follows from (6.2.19) that

$$|Q(Y+Z,\epsilon+\delta)| - |Q(Y,\epsilon)| = D_{Y,\epsilon}|Q|(Z,\delta) + R_{Y,\epsilon}(Z,\delta),$$

with $\|R(Z,\delta)\|_{U^{0,\alpha}} = o(\|(Z,\delta)\|_{U^{2,\alpha}\times I})$. In order to conclude that $D_{Y,\epsilon}|Q|$ is the derivative of $|Q(Y,\epsilon)|$, we only need to check that, it is (i) well-defined (i.e. that its image belongs to $U^{0,\alpha}$) and (ii) that it is bounded, i.e. that $\|D_{Y,\epsilon}|Q|(Z,\delta)\|_{U^{0,\alpha}} < C\|(Z,\delta)\|_{U^{2,\alpha}\times I}$ for some constant C.

To show (i), let us concentrate on the most difficult term which is $-\sigma B_{Y,\epsilon}\dot{Z}$ (because $B_{Y,\epsilon}(x)$ is even and need not vanish at $x=0$). Since $\dot{Z}$ is an odd function, $-\sigma B_{Y,\epsilon}\dot{Z}$ is continuous. To show it is also Hölder continuous, we only need to consider points $x_1 = -a$ and $x_2 = b$ with $0 < a < b$ (if $x_1 \cdot x_2 \geq 0$, the *sign* function remains constant, so $-\sigma B_{Y,\epsilon}\dot{Z}$ is in fact $C^{1,\alpha}$). Calling $w(x) \equiv -\sigma(x)B_{Y,\epsilon}(x)\dot{Z}(x)$ and using that $w(x)$ is even, we find

$$\begin{aligned}
&|w(x_2) - w(x_1)| = |w(b) - w(-a)| = |w(b) - w(a)| = \\
&\left|\left.\frac{d(B_{Y,\epsilon}\dot{Z})}{dx}\right|_{x=\zeta}\right| |b-a| = \left|\left.\frac{d(B_{Y,\epsilon}\dot{Z})}{dx}\right|_{x=\zeta}\right| |b-a|^{1-\alpha}|b-a|^{\alpha} \leq \\
&\left|\left.\frac{d(B_{Y,\epsilon}\dot{Z})}{dx}\right|_{x=\zeta}\right| |b-a|^{1-\alpha}|x_2-x_1|^{\alpha} \leq \left|\left.\frac{d(B_{Y,\epsilon}\dot{Z})}{dx}\right|_{x=\zeta}\right| |x_2-x_1|^{\alpha} \\
&\leq \sup_x \left|\frac{d(B_{Y,\epsilon}\dot{Z})}{dx}\right| |x_2-x_1|^{\alpha}
\end{aligned} \qquad (6.2.20)$$

where the mean value theorem has been applied in the third equality and $\zeta \in (a,b)$. We also have used that $|b-a|^{\alpha} \leq |b+a|^{\alpha} = |x_2-x_1|^{\alpha}$ and $|b-a| < 1$. This proves that $-\sigma B_{Y,\epsilon}\dot{Z}$ is Hölder continuous with exponent α. The remaining terms $-\sigma(x)A_{Y,\epsilon}(x)Z(x)$ and $-\sigma(x)C_{Y,\epsilon}(x)\delta$ are obviously continuous because

they vanish at $x = 0$. To show Hölder continuity the same argument that for $-\sigma(x)B_{Y,\epsilon}(x)\dot{Z}$ works.

To check (ii), we have to find and upper bound for the norm $\|w(x)\|_{U^{0,\alpha}}$.

$$\begin{aligned}
&\|w(x)\|_{U^{0,\alpha}} = \sup_x |w(x)| + \sup_{x_1\neq x_2} \frac{|w(x_2)-w(x_1)|}{|x_2-x_1|^\alpha} \\
&\leq \sup_x |B_{Y,\epsilon}(x)| \sup_x |\dot{Z}(x)| + \sup_x \left|\frac{d(B_{Y,\epsilon}\dot{Z})}{dx}\right| \\
&\leq \sup_x |B_{Y,\epsilon}(x)| \sup_x |\dot{Z}(x)| + \sup_x |\dot{B}_{Y,\epsilon}(x)| \sup_x |\dot{Z}(x)| + \sup_x |B_{Y,\epsilon}(x)| \sup_x |\ddot{Z}(x)| \\
&\leq (2\sup_x |B_{Y,\epsilon}(x)| + \sup_x |\dot{B}_{Y,\epsilon}(x)|)\|(Z,\delta)\|_{U^{2,\alpha}\times I},
\end{aligned}$$

where, in the first inequality, (6.2.20) has been used. Since $B_{Y,\epsilon}(x)$ is $C^{1,\alpha}$, then $(2\sup_x |B_{Y,\epsilon}(x)| + \sup_x |\dot{B}_{Y,\epsilon}(x)|)$ is bounded in the compact set $[-1,1]$ and, therefore, there exists a constant C such that $\|-\sigma B_{Y,\epsilon}\dot{Z}\|_{U^{0,\alpha}} < C\|(Z,\delta)\|_{U^{2,\alpha}\times I}$. A similar argument applies to $-\sigma A_{Y,\epsilon}Z$ and $-\sigma C_{Y,\epsilon}\delta$ and we conclude that $D_{Y,\epsilon}|Q|$ is indeed a continuous operator.

In order to apply the implicit function theorem, it is furthermore necessary that $|Q| \in C^1(U^{2,\alpha}\times I)$ (i.e. that $D_{Y,\epsilon}|Q|$ depends continuously on (Y,ϵ)). This means that given any convergent sequence $(Y_n,\epsilon_n) \in \mathcal{V}_{r_0}$, the corresponding operators $D_{Y_n,\epsilon_n}|Q|$ also converge. Denoting by $(Y,\epsilon) \in \mathcal{V}_{r_0}$ the limit of the sequence, we need to prove that

$$\|D_{Y_n,\epsilon_n}|Q| - D_{Y,\epsilon}|Q|\|_{\pounds(U^{2,\alpha}\times I, U^{0,\alpha})} \to 0,$$

where, for any linear operator $\mathscr{L} : U^{2,\alpha}\times I \to U^{0,\alpha}$, the operator norm is

$$\|\mathscr{L}\|_{\pounds(U^{2,\alpha}\times I, U^{0,\alpha})} \equiv \sup_{(Z,\delta)\neq(0,0)} \frac{\|\mathscr{L}(Z,\delta)\|_{U^{0,\alpha}}}{\|(Z,\delta)\|_{U^{2,\alpha}\times I}}.$$

For that it suffices to find a constant K (which may depend on (Y,ϵ)), such that

$$\begin{aligned}
&\|(D_{Y_n,\epsilon_n}|Q| - D_{Y,\epsilon}|Q|)(Z,\delta)\|_{U^{0,\alpha}} \\
&\qquad \leq K\|(Z,\delta)\|_{U^{2,\alpha}\times I}\|(Y_n - Y, \epsilon_n - \epsilon)\|_{U^{2,\alpha}\times I} \qquad (6.2.21)
\end{aligned}$$

for all $(Z,\delta) \in U^{2,\alpha}\times I$. Indeed, if (6.2.21) holds then the right-hand side tends to zero when $(Y_n,\epsilon_n) \to (Y,\epsilon)$ Again, the most difficult case involves $\sigma(B_{Y,\epsilon} - B_{Y_n,\epsilon_n})\dot{Z}$, so let us concentrate on this term (the same argument works for the remaining terms in $D_{Y_n,\epsilon_n}|Q| - D_{Y,\epsilon}|Q|$).

With the definition $z \equiv \sigma(B_{Y,\epsilon} - B_{Y_n,\epsilon_n})\dot{Z}$, we have

$$\sup_x |z(x)| \leq \sup_x |B_{Y,\epsilon}(x) - B_{Y_n,\epsilon_n}(x)| \sup_x |\dot{Z}(x)|.$$

To bound the C^0-norm of z in terms of $\|(Z,\delta)\|_{U^{2,\alpha}\times I}\|(Y_n - Y, \epsilon_n - \epsilon)\|_{U^{2,\alpha}\times I}$, we have to use the mean value theorem on the function $\mathcal{B} \equiv \partial_2 \mathcal{Q}$ (recall that $B_{Y,\epsilon}(x) = \mathcal{B}|_{(Y(x),\dot{Y}(x),x,\epsilon)}$). By the definition of $\mathcal{V}_{r_0}$ (see (6.2.18)) any element $(Y,\epsilon) \in \mathcal{V}_{r_0}$ satisfies that $|Y - Y_1|(x) \le r_0$ and $|\dot{Y} - \dot{Y}_1|(x) \le r_0$ $\forall x \in [-1,1]$. This implies that there is a compact set $\mathbb{K} \subset \mathbb{R}^4$ depending only on r_0 and Y_1 such that $(Y(x), \dot{Y}(x), x, \epsilon) \in \mathbb{K}$, for all $x \in [-1,1]$ and $(Y,\epsilon) \in \mathcal{V}_{r_0}$. When applying the mean value theorem to the derivatives $\partial_1\mathcal{B}$, $\partial_2\mathcal{B}$ and $\partial_4\mathcal{B}$ all mean value points will therefore belong to $\mathbb{K}$. Taking the supremum of these derivatives in $\mathbb{K}$, we get the following bound.

$$\sup_x |z(x)| \le \sup_{\mathbb{K}} (|\partial_1\mathcal{B}| + |\partial_2\mathcal{B}| + |\partial_4\mathcal{B}|) \sup_x |\dot{Z}| \|(Y_n - Y, \epsilon_n - \epsilon)\|_{U^{2,\alpha}\times I}. \quad (6.2.22)$$

Since $\mathcal{B}$ is smooth, (6.2.22) is already of the form (6.2.21).

It only remains to bound the Hölder norm of z in a similar way. As before, this is done by distinguishing two cases, namely when $x_1 \cdot x_2 \ge 0$ and when $x_1 \cdot x_2 < 0$. If $x_1 \cdot x_2 \ge 0$ then $\sigma(x)$ is a constant function and therefore, to obtaining an inequality of the form

$$\sup_{x_1 \neq x_2} \frac{|z(x_2) - z(x_1)|}{|x_2 - x_1|^\alpha} \le K_1 \|(Z,\delta)\|_{U^{2,\alpha}\times I}\|(Y_n - Y, \epsilon_n - \epsilon)\|_{U^{2,\alpha}\times I}$$

is standard (and a consequence of Theorem B.3). When $x_1 \cdot x_2 < 0$, we exploit the parity of the functions as in (6.2.20) to get

$$|z(x_2) - z(x_1)| \le \left| \left. \frac{d((B_{Y_n,\epsilon_n} - B_{Y,\epsilon})\dot{Z})}{dx} \right|_{x=\zeta} \right| |x_2 - x_1|^\alpha,$$

where $\zeta \in (a,b)$ and we are assuming $x_1 = -a, x_2 = b, 0 < a < b$ without loss of generality. Since the sign function $\sigma(x)$ has already disappeared, a bound for the right hand side in terms of $K_2\|(Z,\delta)\|_{U^{2,\alpha}\times I}\|(Y_n - Y, \epsilon_n - \epsilon)\|_{U^{2,\alpha}\times I}|x_2 - x_1|^\alpha$ is guaranteed by Theorem B.3. This, combined with (6.2.22) gives (6.2.21) and hence continuity of the derivative of $D_{Y,\epsilon}|Q|$ with respect to $(Y,\epsilon) \in \mathcal{V}_{r_0}$.

The final requirement to apply the implicit function theorem to $F = P - |Q|$ is to check that $D_Y F|_{(Y_1,\epsilon=0)}$ is an isomorphism between $U^{2,\alpha}$ and $U^{0,\alpha}$. This is immediate from equation (6.2.17) that implies

$$D_Y F|_{(Y_1,\epsilon=0)}(Z) = F(Y_1 + Z, \epsilon = 0) - F(Y_1, \epsilon = 0) = cL(Z),$$

and we have already shown that L is an isomorphism.

Thus, the implicit function theorem can be used to conclude that there exists an open neighbourhood $\tilde{I} \subset I$ of $\epsilon = 0$ and a C^1 map $\tilde{Y} : \tilde{I} \to U^{2,\alpha}$ such

that $\tilde{Y}(\epsilon = 0) = Y_1$ and $y = \epsilon \tilde{Y}(\epsilon)$ defines a $C^{2,\alpha}$ generalized apparent horizon embedded in Σ_ϵ.

■

We will denote by $\hat{S}_\epsilon$ the surface defined by this solution. The proposition above implies that we can expand $y(x, \epsilon) = Y_1(x)\epsilon + o(\epsilon)$. From (6.2.16) it follows that Y_1 satisfies the linear equation $L[Y_1(x)] = 3|x|$. Decomposing $Y_1(x)$ into Legendre polynomials $P_l(x)$, as $Y_1(x) = \sum_{l=0}^{\infty} a_l P_l(x)$, where convergence is in $L^2[-1, 1]$, this equation reads

$$L[Y_1(x)] = \sum_{l=0}^{\infty} a_l L[P_l(x)] = 3|x|.$$

The Legendre equation, $-(1 - x^2)\ddot{P}_l(x) + 2x\dot{P}_l(x) - l(l+1)P_l(x) = 0$, implies that $L[P_l(x)] = (l(l + 1) + 1)P_l(x)$. We can also decompose $|x|$ in terms of Legendre polynomials. This computation can be found in [18] and gives

$$|x| = \frac{1}{2} + \sum_{l=1}^{\infty} b_{2l} P_{2l}(x),$$

where

$$b_{2l} = \frac{(4l + 1)(-1)^{l+1}}{2^{2l}} \frac{(2l - 2)!}{(l - 1)!(l + 1)!}, \qquad l \geq 1.$$

It follows that the unique solution to the equation $L[Y_1(x)] = 3|x|$ is

$$Y_1(x) = \frac{3}{2} + \sum_{l=1}^{\infty} a_{2l} P_{2l}(x), \tag{6.2.23}$$

with

$$a_{2l} = \frac{3(4l + 1)(-1)^{l+1}}{[2l(2l + 1) + 1]\, 2^{2l}} \frac{(2l - 2)!}{(l - 1)!(l + 1)!}, \qquad l \geq 1 \tag{6.2.24}$$

(see Figure 6.1).

6.2.2 Area of the outermost generalized trapped horizon

In this subsection we will compute the area of $\hat{S}_\epsilon$, to second order in ϵ, and we will obtain that it is greater than $16\pi M_{Kr}^2$. Then, we will prove that any generalized apparent horizon enclosing $\hat{S}_\epsilon$ has greater or equal area than $\hat{S}_\epsilon$ which will complete the proof of Theorem 6.1.1.

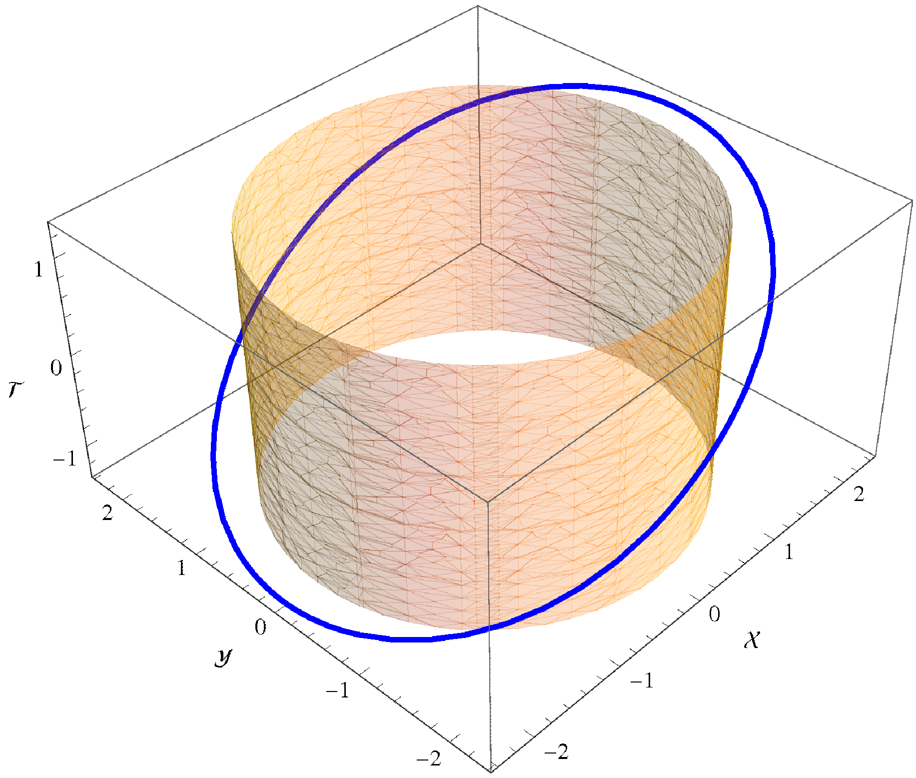


Figure 6.1: Parametic plot of the solution $Y_1(\cos\theta)$ (in blue) in coordinates $\mathcal{T} \equiv M_{Kr} \ln \frac{\hat{v}}{\bar{v}}$, $\mathcal{X} = r\cos\theta$ and $\mathcal{Y} = r\sin\theta$ where θ has been allowed to vary between 0 and 2π, $M_{Kr} = 1$ and $\epsilon = 0.5$. The figure also shows the set $\{r = 2M\}$ (in gold) in these coordinates. Note that the solution lies entirely outside the region $\{r \leq 2M\}$ (i.e. the region inside the cylinder).

Integrating the volume element of $\hat{S}_\epsilon$, it is straightforward to get

$$
\begin{aligned}
|\hat{S}_\epsilon| &= \int_{-1}^{1}\int_{0}^{2\pi} r^2\sqrt{1+\epsilon^2\frac{32M_{Kr}^3}{r^3}e^{-r/2M_{Kr}}(1-x^2)(\dot{Y}_1^2-1)+O(\epsilon^3)}d\phi dx \\
&= \int_{-1}^{1}\int_{0}^{2\pi}\left[r^2+\epsilon^2\frac{16M_{Kr}^3}{r}e^{-r/2M_{Kr}}(1-x^2)(\dot{Y}_1^2-1)+O(\epsilon^3)\right]d\phi dx,
\end{aligned}
$$

where r still depends on ϵ. Let us expand $r = r_0 + r_1\epsilon + r_2\epsilon^2 + O(\epsilon^3)$. Using equation (6.2.2) and expanding the exponential therein, it follows

$$
r = 2M_{Kr} + \frac{2M_{Kr}}{e}(Y_1^2 - x^2)\epsilon^2 + O(\epsilon^3). \qquad (6.2.25)
$$

Then, after inserting (6.2.23), (6.2.24) and (6.2.25) into the integral and using the orthogonality properties of the Legendre polynomials, we find

$$
|\hat{S}_\epsilon| = 16\pi M_{Kr}^2 + \frac{8\pi M_{Kr}^2\epsilon^2}{e}\left(5+4\sum_{l=1}^{\infty}\frac{2l(2l+1)+1}{4l+1}a_{2l}^2\right)+O(\epsilon^3).
$$

Since the second term is strictly positive, it follows that $|\hat{S}_\epsilon| > 16\pi M_{Kr}^2$. This is not yet a counterexample of (6.1.1) because $\hat{S}_\epsilon$ is not known to be the outermost generalized apparent horizon. Before turning into this point, however, let us give an alternative argument to show that the area increases. This will shed some light into the underlying reason why the area of $\hat{S}_\epsilon$ is larger than $16\pi M_{Kr}^2$.

To that aim, let us now use coordinates $\{\hat{u}, x, \phi\}$ in Σ_ϵ. Then, the embedding of Σ_ϵ becomes $\Sigma_\epsilon \equiv \{\hat{u}, \hat{v} = \hat{u} + 2\epsilon x, x, \phi\}$, and the corresponding embedding in Σ_ϵ for the surfaces $\hat{S}_\epsilon$ is $\hat{S}_\epsilon = \{\hat{u} = u(x,\epsilon), x, \phi\}$. Again, u admits an expansion $u = U_1(x)\epsilon + o(\epsilon)$. The relationship between U_1 and Y_1 is simply $Y_1 = U_1 + x$. It follows that U_1 satisfies $L[U_1(x)] = 3(|x| - x)$. Similarly, if we take $\{\hat{v}, x, \phi\}$ as coordinates for Σ_ϵ, then the embedding of $\hat{S}_\epsilon$ reads $\hat{v} = V_1(x)\epsilon + o(\epsilon)$, with V_1 satisfying $Y_1 = V_1 - x$ and therefore $L[V_1(x)] = 3(|x| + x)$. Thus, $L[U_1(x)] \geq 0$ and $L[V_1(x)] \geq 0$ and neither of them is identically zero. Since L is an elliptic operator with positive zero order term, we can use the maximum principle to conclude that $U_1(x) > 0$ and $V_1(x) > 0$ everywhere. Geometrically, this means that $\hat{S}_\epsilon$ lies fully in Σ_ϵ^+ for ϵ small enough (c.f. Figure 6.1). In fact, the maximum principle applied to $L[Y_1] = 3|x|$ also implies $Y_1 > 0$. This will be used below.

We can now view $\hat{S}_\epsilon$ as a first order spacetime variation of the bifurcation surface $\hat{S}_{\epsilon=0}$. The variation vector ∂_ϵ is defined as the tangent vector to the curve generated when a point with fixed coordinates $\{x, \phi\}$ in $\hat{S}_\epsilon$ moves as ϵ varies. This vector satisfies $\partial_\epsilon = U_1\partial_{\hat{u}} + V_1\partial_{\hat{v}} + O(\epsilon)$ and is spacelike everywhere

on the unperturbed surface $\hat{S}_{\epsilon=0}$. If we do a Taylor expansion of $|\hat{S}_\epsilon|$ around $\epsilon = 0$, we see that the zero order term is $|\hat{S}_{\epsilon=0}| = 16\pi M_{Kr}^2$, as this is the area of the bifurcation surface. The bifurcation surface is totally geodesic so that, in particular, its mean curvature vector vanishes. Consequently, the linear term in the expansion is identically zero as a consequence of the first variation of area (2.2.3). For any $\epsilon \geq 0$ we have

$$\begin{aligned}\frac{d|\hat{S}_\epsilon|}{d\epsilon} &= \int_{\hat{S}_\epsilon} (\vec{H}_{\hat{S}_\epsilon}, \partial_\epsilon)\boldsymbol{\eta}_{\hat{\boldsymbol{S}}_\epsilon} \\ &= \int_{\hat{S}_\epsilon} \left(-\frac{1}{2}\left[(p+q)\vec{l}_- + (-p+q)\vec{l}_+ \right], U_1\partial_{\hat{u}} + V_1\partial_{\hat{v}} + O(\epsilon) \right) \boldsymbol{\eta}_{\hat{\boldsymbol{S}}_\epsilon} \quad (6.2.26)\end{aligned}$$

where $\vec{H}_{\hat{S}_\epsilon}$ is the spacetime mean curvature vector of $\hat{S}_\epsilon$, $(\ ,\ )$ denotes the scalar product with the spacetime metric, and $\vec{l}_+$ and $\vec{l}_-$ are the outer and the inner null vectors which are future directed and satisfy $(\vec{l}_+, \vec{l}_-) = -2$. Since on $\hat{S}_{\epsilon=0}$ the vectors $\partial_{\hat{v}}$ and $-\partial_{\hat{u}}$ are proportional to $\vec{l}_+$ and $\vec{l}_-$, we have

$$\begin{aligned}\vec{l}_+|_{\hat{S}_\epsilon} &= \sqrt{\frac{e}{8M_{Kr}^2}}\partial_{\hat{v}} + O(\epsilon), \\ \vec{l}_-|_{\hat{S}_\epsilon} &= \sqrt{\frac{e}{8M_{Kr}^2}}(-\partial_{\hat{u}}) + O(\epsilon),\end{aligned}$$

where the factor $\sqrt{\frac{e}{8M_{Kr}^2}}$ is due to the normalization $(l_+, l_-) = -2$. Besides, $\boldsymbol{\eta}_{\hat{\boldsymbol{S}}_\epsilon} = 4M_{Kr}^2 dx \wedge d\phi + O(\epsilon)$. Then, inserting these expressions into the first variation integral (6.2.26) and taking the derivative with respect to ϵ at $\epsilon = 0$, we obtain

$$\left.\frac{d^2|\hat{S}_\epsilon|}{d\epsilon^2}\right|_{\epsilon=0} = \frac{16\sqrt{2}\pi M_{Kr}^2}{e}\int_{-1}^{1}\Big[U_1(x)L[V_1(x)] + V_1(x)L[U_1(x)] \Big]\, dx,$$

where (6.2.3), (6.2.4) and the relations $Y_1 = U_1 + x$ and $Y_1 = V_1 - x$ has been used. Since U_1 and V_1 are strictly positive and $L[U_1(x)]$, $L[V_1(x)]$ are non-negative and not identically zero, it follows $\left.\frac{d^2|\hat{S}_\epsilon|}{d\epsilon^2}\right|_{\epsilon=0} > 0$ and hence that the area of $\hat{S}_\epsilon$ is larger than $16\pi M_{Kr}^2$ for small ϵ.

We have obtained that the second order variation of area turns out to be strictly positive along the direction joining the bifurcation surface with $\hat{S}_\epsilon$, which is tied to the fact that $L[U_1]$ and $L[V_1]$ have a sign. The right hand sides of these operators are (except for a constant) the linearization of $|q| \pm q$ and these objects are obviously non-negative in all cases. We conclude, therefore, that the fact that the area of $\hat{S}_\epsilon$ is larger than $16\pi M_{Kr}^2$ is closely related to the defining equation $p = |q|$. It follows that the increase of area is a robust property which does not

depend strongly on the choice of hypersurfaces Σ_ϵ that we have made. In fact, had we chosen hypersurfaces $\Sigma_\epsilon \equiv \{u = y - \epsilon\beta(x), v = y + \epsilon\beta(x), \cos\theta = x, \phi = \phi\}$, the corresponding equations would have been $L[U_1(x)] = |L[\beta(x)]| - L[\beta(x)]$ and $L[V_1(x)] = |L[\beta(x)]| + L[\beta(x)]$. The same conclusions would follow provided the right hand sides are not identically zero.

Having shown that $|\hat{S}_\epsilon| > 16\pi M_{Kr}^2$ for $\epsilon \neq 0$ small enough, the next step is to analyze whether $|\hat{S}_\epsilon|$ is a lower bound for the area of the outermost generalized apparent horizon. Indeed, in order to have a counterexample of (6.1.1) we only need to make sure that no generalized apparent horizon with less area than $\hat{S}_\epsilon$ and enclosing $\hat{S}_\epsilon$ exists in Σ_ϵ.

We will argue by contradiction. Let S'_ϵ be a generalized apparent horizon enclosing $\hat{S}_\epsilon$ and with $|S'_\epsilon| < |\hat{S}_\epsilon|$. In these circumstances, $\hat{S}_\epsilon$ cannot be area outer minimizing. Thus, its minimal area enclosure $\hat{S}'_\epsilon$ does not coincide with it. Now, two possibilities arise: (i) either $\hat{S}'_\epsilon$ lies completely outside $\hat{S}_\epsilon$, or (ii) it coincides with $\hat{S}_\epsilon$ on a closed subset $\mathcal{K}$, while the complement $\hat{S}'_\epsilon \setminus \mathcal{K}$ (which is non-empty) has vanishing mean curvature p everywhere.

To exclude case (i), consider the foliation of Σ_ϵ defined by the surfaces $\{\hat{y} = y_0, x, \phi\}$, where y_0 is a constant. We then compute the mean curvature p_{y_0} of these surfaces. The induced metric is

$$\gamma_{AB}^{y_0} = \left(\frac{r^2}{1-x^2} - \epsilon^2 \frac{32M_{Kr}^3}{r} e^{-r/2M_{Kr}} \right) dx^2 + (1-x^2) r^2 d\phi^2.$$

The tangent vectors and the unit normal one-form are

$$\vec{e}_x = \partial_x, \quad \vec{e}_\phi = \partial_\phi, \quad \mathbf{m} = A d\hat{y},$$

where $A = \sqrt{\frac{32M_{Kr}^3}{r} e^{-r/2M_{Kr}}}$ is the normalization factor. Since $\gamma^{\hat{y}_0}$ is diagonal we just need the following derivatives

$$\begin{aligned} \nabla^{\Sigma_\epsilon}_{\vec{e}_x} e^{\hat{y}}_x &= -\frac{r^3 + 8\epsilon^2 M_{Kr}^2 (2M_{Kr} + r)(1-x^2) e^{-r/2M_{Kr}}}{4M_{Kr}(1-x^2)r^2} y_0 \\ \nabla^{\Sigma_\epsilon}_{\vec{e}_\phi} e^{\hat{y}}_\phi &= -\frac{(1-x^2)r}{4M_{Kr}} y_0. \end{aligned}$$

Inserting all these expressions in $p_{y_0} = -m_i \gamma^{AB} \nabla^{\Sigma_\epsilon}_{\vec{e}_A} e^i_B$ we obtain

$$p_{y_0} = A \left(\frac{r^3 + 8\epsilon^2 M_{Kr}^2 (2M_{Kr} + r)(1-x^2) e^{-r/2M_{Kr}}}{4M_{Kr} r \left(r^3 - 32\epsilon^2 M_{Kr}^3 (1-x^2) e^{-r/2M_{Kr}} \right)} + \frac{1}{4M_{Kr} r} \right) y_0.$$

Thus, taking $-1 < \epsilon < 1$ small enough so that

$$\epsilon^2 < \frac{r_{\min}^3 e^{r_{\min}/2M_{Kr}}}{32M_{Kr}^3},$$

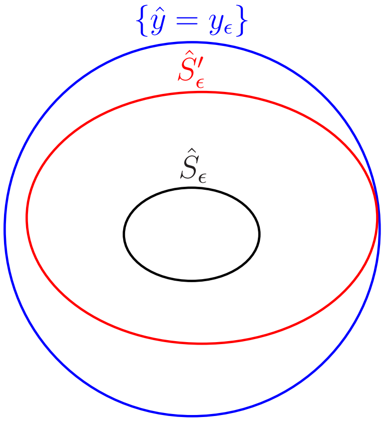


Figure 6.2: If the minimal area enclosure $\hat{S}'_\epsilon$ (in red) lies completely outside $\hat{S}_\epsilon$ then $\hat{S}'_\epsilon$, which is a minimal surface, must touch tangentially from the inside a surface $\{\hat{y} = y_\epsilon\}$ (in blue) which has $p_{y_\epsilon} > 0$.

where $r_{\min}$ is the minimum value of r in Σ_ϵ (recall that $r_{\min} > 0$ provided $|\epsilon| < 1$), we can assert that $p_{y_0} > 0$ for all $y_0 > 0$.

We noted above that $Y_1(x) > 0$ everywhere. Thus, for small enough positive ϵ, the function $y(x, \epsilon)$ is also strictly positive. Since $\hat{S}'_\epsilon$ lies fully outside $\hat{S}_\epsilon$, the coordinate function $\hat{y}$ restricted to $\hat{S}'_\epsilon$ achieves a positive maximum y_ϵ somewhere. At this point, the two surfaces $\hat{S}'_\epsilon$ and $\{\hat{y} = y_\epsilon\}$ meet tangentially, with $\hat{S}'_\epsilon$ lying fully inside $\{\hat{y} = y_\epsilon\}$ (see Figure 6.2). This is a contradiction to the maximum principle for minimal surfaces (see Proposition B.7 with $K = 0$ in Appendix B).

It only remains to deal with case (ii). The same argument above shows that the coordinate function $\hat{y}$ restricted to $\hat{S}'_\epsilon \setminus \mathcal{K}$ cannot reach a local maximum. It follows that the range of variation of $\hat{y}$ restricted to $\hat{S}'_\epsilon$ is contained in the range of variation of $\hat{y}$ restricted to $\hat{S}_\epsilon$ (see Figure 6.3).

Since $\max_{\hat{S}_\epsilon} \hat{y} - \min_{\hat{S}_\epsilon} \hat{y} = O(\epsilon)$, it follows that we can regard $\hat{S}'_\epsilon$ as an outward variation of $\hat{S}_\epsilon$ of order ϵ when ϵ is taken small enough. The corresponding variation vector field $\vec{\nu}$ can be taken orthogonal to $\hat{S}_\epsilon$ without loss of generality, i.e. $\vec{\nu} = \nu \vec{m}$, where $\vec{m}$ is the outward unit normal to $\hat{S}_\epsilon$. The function ν vanishes on $\mathcal{K}$ and is positive in its complement $U \equiv \hat{S}_\epsilon \setminus \mathcal{K}$. Expanding to second order and using the first and second variation of area (see e.g. [36]) gives

$$\begin{aligned} |\hat{S}'_\epsilon| &= |\hat{S}_\epsilon| + \epsilon \int_U p_{\hat{S}_\epsilon} \nu \boldsymbol{\eta}_{\hat{\boldsymbol{S}}_\epsilon} \\ &\quad + \frac{\epsilon^2}{2} \int_U \left(|\nabla_{\hat{S}_\epsilon} \nu|^2 + \frac{\nu^2}{2} \left(R^{\hat{S}_\epsilon} - R^{\Sigma_\epsilon} - |\kappa_{\hat{S}_\epsilon}|^2 + p^2_{\hat{S}_\epsilon} \right) + p_{\hat{S}_\epsilon} \frac{d\nu}{d\epsilon} \right) \boldsymbol{\eta}_{\hat{\boldsymbol{S}}_\epsilon} + O(\epsilon^3), \end{aligned}$$

where $\nabla_{\hat{S}_\epsilon}$, $R^{\hat{S}_\epsilon}$ and $\kappa_{\hat{S}_\epsilon}$ are, respectively, the gradient, scalar curvature and second

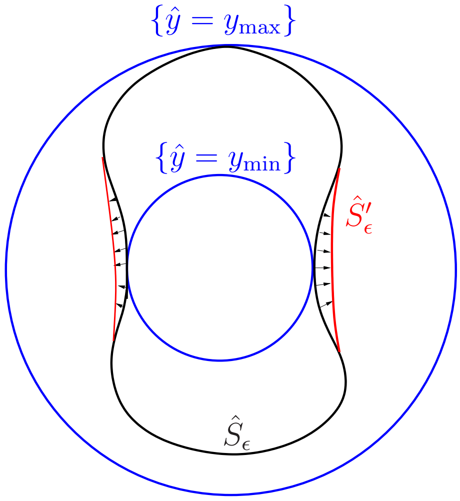


Figure 6.3: In the case (ii), the minimal area enclosure $\hat{S}'_\epsilon$ coincides with $\hat{S}_\epsilon$ in a compact set. The coordinate function $\hat{y}$ restricted to $\hat{S}'_\epsilon$ cannot achieve a local maximum in the set where $\hat{S}'_\epsilon$ and $\hat{S}_\epsilon$ do not coincide (in red). Then, this set can be viewed as an outward variation of order ϵ of the corresponding points in $\hat{S}_\epsilon$.

fundamental form of $\hat{S}_\epsilon$, and R^{Σ_ϵ} is the scalar curvature of Σ_ϵ. Now, the mean curvature $p_{\hat{S}_\epsilon}$ of $\hat{S}_\epsilon$ reads $p_{\hat{S}_\epsilon} = \frac{3\epsilon}{M_{Kr}\sqrt{e}}|x| + o(\epsilon)$ (see equation (6.2.3)) and both R^{Σ_ϵ} and $\kappa_{\hat{S}_\epsilon}$ are of order ϵ (because $\Sigma_{\epsilon=0}$ has vanishing scalar curvature and $\hat{S}_{\epsilon=0}$ is totally geodesic). Moreover, $R^{\hat{S}_\epsilon} = 1/(2M_{Kr}^2) + O(\epsilon)$. Thus,

$$|\hat{S}'_\epsilon| = |\hat{S}_\epsilon| + \epsilon^2 \left\{ \int_U \left[\frac{3|x|\nu}{M_{Kr}\sqrt{e}} + \left(\frac{|\nabla_{\hat{S}_\epsilon}\nu|^2}{2} + \frac{\nu^2}{8M_{Kr}^2} \right) \right] \boldsymbol{\eta}_{\hat{\boldsymbol{S}}_\epsilon} \right\} + O(\epsilon^3).$$

It follows that, for small enough ϵ, the area of $\hat{S}'_\epsilon$ is larger than $\hat{S}_\epsilon$ contrarily to our assumption. This proves Theorem 6.1.1 and, therefore, the existence of counterexamples to the version (6.1.1) of the Penrose inequality.

It is important to remark that the existence of this counterexample does not invalidate the approach suggested by Bray and Khuri to study the general Penrose inequality. It means, however, that the emphasis should not be put on generalized apparent horizons. It may be that the approach can serve to prove the standard version (2.3.6) as recently discussed in [21].

Chapter 7
Conclusions

In this thesis we have studied some questions within the framework of the theory of General Relativity. In particular, we have concentrated on some of the properties of marginally outer trapped surfaces (MOTS) and weakly outer trapped surfaces in spacetimes with symmetries, specially static isometries, and its application to the uniqueness theorems of black holes and the Penrose inequality. We can summarize the main results of this thesis in the following list.

1. We have obtained a general expression for the first variation of the outer null expansion θ^+ of a surface S along an arbitrary vector field $\vec{\xi}$ in terms of the deformation tensor of the spacetime metric associated with the vector $\vec{\xi}$. This expression has been particularized when S is a MOTS.

2. Starting from a geometrical idea that generates a family of surfaces by moving first along $\vec{\xi}$ and then along null geodesics, we have used the theory of linear elliptic second order operators to obtain restrictions on any vector field on stable and strictly stable MOTS. Using the expression mentioned in the previous point, these results have been particularized to generators of symmetries of physical interest, such as Killing vectors, homotheties and conformal Killing vectors. As an application we have shown that there exists no stable MOTS in any spacelike hypersurface of a large class of Friedmann-Lemaître-Robertson-Walker cosmological models, which includes all classic models of matter and radiation dominated eras and those models with accelerated expansion which satisfy the null energy condition (NEC).

3. For the situations when the elliptic theory is not useful, we have exploited the geometrical idea mentioned before to obtain similar restrictions for Killing vectors and homotheties on outermost and locally outermost MOTS. As a consequence of these results, we have shown that, on a spacelike hyper-

surface possessing an untrapped barrier S_b, a Killing vector or a homothety $\vec{\xi}$ cannot be timelike anywhere on a bounding weakly outer trapped surface whose exterior lies in the region where $\vec{\xi}$ is timelike, provided the NEC holds in the spacetime.

For the more general cases when the elliptic theory simply cannot be applied, a suitable variation of the geometrical idea has allowed us to obtain weaker restrictions on any vector field $\vec{\xi}$ on locally outermost MOTS. This results have also been particularized to Killing vectors, homotheties and conformal Killing vectors.

4. Analyzing the Killing form in a static Killing initial data (KID) $(\Sigma, g, K; N, \vec{Y}, \tau)$ we have shown, at the initial data level, that the topological boundary of each connected component $\{\lambda > 0\}_0$ of the region where the Killing vector is timelike is a smooth injectively immersed submanifold with $\theta^+ = 0$ with respect to the outer normal which points into $\{\lambda > 0\}_0$, provided

   (i) $NY^i \nabla^\Sigma_i \lambda|_{\partial^{top}\{\lambda>0\}_0} \geq 0$ if $\partial^{top}\{\lambda > 0\}_0$ contains at least one fixed point.

   (ii) $NY^i m_i|_{\partial^{top}\{\lambda>0\}_0} \geq 0$ if $\partial^{top}\{\lambda > 0\}_0$ contains no fixed point, where $\vec{m}$ is the unit normal pointing towards $\{\lambda > 0\}_0$.

   There are examples in the Kruskal spacetime where these conditions do not hold and $\partial^{top}\{\lambda > 0\}_0$ fails to be smooth and has $\theta^+ \neq 0$.

5. Under the same hypotheses as before we have proven a confinement result for MOTS in arbitrary spacetimes satisfying the NEC and for arbitrary spacelike hypersurfaces, not necessarily time-symmetric. The hypersurfaces need not be asymptotically flat either and are only required to have an outer untrapped barrier S_b. This result, which also have been proved at the initial data level, asserts that no bounding weakly outer trapped surface can intersect $\{\lambda > 0\}^{ext}$, where $\{\lambda > 0\}^{ext}$ denotes the connected component of $\{\lambda > 0\}$ which contains S_b. A condition which ensures that all arc-connected components of $\partial^{top}\{\lambda > 0\}$ are topologically closed is required. This condition is automatically fulfilled in spacetimes containing no non-embedded Killing prehorizons.

6. We have proven that the set $\partial^{top}\{\lambda > 0\}$ in an embedded static KID is a union of smooth injectively immersed surfaces with at least one of the two

null expansions equal to zero (provided the topological condition mentioned in the previous point is satisfied).

7. Using the previous result, we have shown that, in a static embedded KID which satisfies the NEC and possesses an outer untrapped barrier S_b and a bounding weakly outer trapped surface, the set $\partial^{top}\{\lambda > 0\}^{ext}$ is the outermost bounding MOTS provided that every arc-connected component of $\partial^{top}\{\lambda > 0\}^{ext}$ is topologically closed, the past weakly outer trapped region T^- is contained in the weakly outer trapped region T^+ and a topological condition which ensures that all closed orientable surfaces separate the manifold.

8. With the previous result at hand, we have obtained a uniqueness theorem for embedded static KID containing an asymptotically flat end which satisfy the NEC and possess a bounding weakly outer trapped surface. The matter model is arbitrary as long as it admits a static black hole uniqueness proof with the Bunting and Masood-ul-Alam doubling method. This result extends a previous theorem by Miao valid on vacuum and time-symmetric slices, and allows to conclude that, at least regarding uniqueness of black holes, event horizons and MOTS do coincide in static spacetimes. This result requires the same hypotheses as the result in the previous point. As we have mentioned before, the condition on the arc-connected components of $\partial^{top}\{\lambda > 0\}^{ext}$ is closely related with the non-existence of non-embedded Killing prehorizons and can be removed if a result on the non-existence of these type of prehorizons is found. The condition $T^- \subset T^+$ is needed for out argument to work. Trying to drop this hypotheses is a logical next step, but it would require a different method of proof.

9. Finally, we have proved that there exist slices in the Kruskal spacetime where the outermost generalized apparent horizon has area greater than $16\pi M_{Kr}^2$, where M_{Kr} is the mass of the Kruskal spacetime. This gives a counterexample of a Penrose inequality recently proposed by Bray and Khuri (in terms of the area of the outermost apparent horizon) in order to address the general proof of the standard Penrose inequality. The existence of this counterexample does not invalidate the approach of these authors but indicate that the emphasis must not be on generalized apparent horizons.

Appendix A
Differential manifolds

In this Appendix, we will give a definition of a differentiable manifold which allows us to consider manifolds with and without boundary at the same time. We follow [68].

Consider the vector space $\mathbb{R}^n$ and let ω_α be a one-form defined on this vector space (the index α is simply a label at this point). Let us define the set $H_\alpha = \{\vec{r} \in \mathbb{R}^n : \omega_\alpha(\vec{r}) \geq 0\}$, which is either a half plane if $\omega_\alpha \neq 0$ or the whole space if $\omega_\alpha = 0$. The concept of differentiable manifold may be defined as follows.

Definition A.1 *A* **differentiable manifold** *is a topological space M together with a collection of open sets $U_\alpha \subset M$ such that:*

1. *The collection $\{U_\alpha\}$ is an open cover of M, i.e. $M = \bigcup_\alpha U_\alpha$.*

2. *For each α there is a bijective map $\varphi_\alpha : U_\alpha \to V_\alpha$, where V_α is an open subset of H_α with the induced topology of $\mathbb{R}^n$. Every set $(U_\alpha, \varphi_\alpha)$ is called a chart or a local coordinate system. The collection $\{(U_\alpha, \varphi_\alpha)\}$ is called an atlas.*

3. *Consider two sets U_α and U_β which overlap, i.e. $U_\alpha \cap U_\beta \neq \emptyset$, and consider the map $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$. Then, there exists a map $\varphi_{\alpha\beta} : W_\alpha \to W_\beta$, where W_α and W_β are open subsets of $\mathbb{R}^n$ which, respectively, contain $\varphi_\alpha(U_\alpha \cap U_\beta)$ and $\varphi_\beta(U_\alpha \cap U_\beta)$ such that $\varphi_{\alpha\beta}$ is a differentiable bijection, with differentiable inverse and satisfying $\varphi_{\alpha\beta}|_{\varphi_\alpha(U_\alpha \cap U_\beta)} = \varphi_\beta \circ \varphi_\alpha^{-1}$.*

Remark. Since no confusion arises, we will denote a differential manifold $(M, \{U_\alpha\})$ simply by M. Note that manifolds need not be connected according to this definition. □

Definition A.2 *A differentiable manifold M is* **of class** C^k *if the mappings $\varphi_{\alpha\beta}$ and their inverses are C^k.*
A differentiable manifold M is **smooth** *(or C^∞) if it is C^k for all $k \in \mathbb{N}$.*

Definition A.3 *M is a* **differentiable manifold with boundary** *if for at least one chart U_α, we have $\omega_\alpha \neq 0$. In this case, the* **boundary** *of M is defined as $\partial M = \bigcup_{\alpha, \omega_\alpha \neq 0} \{\mathfrak{p} \in U_\alpha$ such that $\omega_\alpha(\varphi_\alpha(\mathfrak{p})) = 0\}$*

Remark. Along this thesis the sign ∂ will denote the boundary of a manifold while the sign ∂^{top} will refer to the *topological* boundary of any subset of a topological space (both concepts are in general completely different). □

Definition A.4 *M is a* **differentiable manifold without boundary** *if $\omega_\alpha = 0$ for all α.*

It can be proven that ∂M is a differentiable manifold without boundary.

Definition A.5 *The interior $int(M)$ of a manifold M is defined as $int(M) = M \setminus \partial M$.*

We will denote by $\overline{U}$ the topological closure of a set U and by $\mathring{U}$ its topological interior.

Definition A.6 *A differentiable manifold, with or without boundary, is* **orientable** *if there exists an atlas such that for any two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) which overlap, i.e. $U_\alpha \cap U_\beta \neq 0$, the Jacobian of $\varphi_{\alpha\beta}|_{U_\alpha \cap U_\beta}$ on $U_\alpha \cap U_\beta$ is positive. Such an atlas will be called* **oriented atlas**
A differentiable manifold with an oriented atlas is said to be **oriented**.

Definition A.7 *Consider an oriented manifold M endowed with a metric $g^{(n)}$. The* **volume element** *$\eta^{(n)}$ of $(M, g^{(n)})$ is the n-form $\eta^{(n)}_{\alpha_1 \ldots \alpha_n} = \sqrt{|det\ g^{(n)}|}\epsilon_{\alpha_1 \ldots \alpha_n}$ in any coordinate chart of the oriented atlas. Here, $\epsilon_{\alpha_1 \ldots \alpha_n}$ is the totally antisymmetric symbol and det $g^{(n)}$ is the determinant of $g^{(n)}$ in this chart.*

All manifolds in thesis are assumed to be Hausdorff and paracompact. These concepts are defined as follows.

Definition A.8 *A topological space M is* **Hausdorff** *if for each pair of points $\mathfrak{p}, \mathfrak{q}$ with $\mathfrak{p} \neq \mathfrak{q}$, there exist two disjoint open sets $U_\mathfrak{p}$ and $U_\mathfrak{q}$ such that $\mathfrak{p} \in U_\mathfrak{p}$ and $\mathfrak{q} \in U_\mathfrak{q}$.*

Definition A.9 *Let M be a topological space and let $\{U_\alpha\}$ be an open cover of M. An open cover $\{V_\beta\}$ is said to be a refinement of $\{U_\alpha\}$ if for each V_β there exists an U_α such that $V_\beta \subset U_\alpha$. The cover $\{V_\beta\}$ is said to be locally finite if each $\mathfrak{p} \in M$ has an open neighbourhood W such that only finitely many V_β satisfy $W \cap V_\beta \neq \emptyset$.*
The topological space M is said to be **paracompact** *if every open cover $\{U_\alpha\}$ of M has a locally finite refinement $\{V_\beta\}$.*

Appendix B

Elements of mathematical analysis

This Appendix is devoted to introducing some elements of mathematical analysis which are used throughout this thesis.

Firstly, recall that a Banach space is a normed vector space which is complete. Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces with respective norms $||\cdot||_{\mathcal{X}}$ and $||\cdot||_{\mathcal{Y}}$. Let $U_{\mathcal{X}} \subset \mathcal{X}$, $U_{\mathcal{Y}} \subset \mathcal{Y}$ be open sets. A function $f : U_{\mathcal{X}} \to U_{\mathcal{Y}}$ is said to be Fréchet-differentiable at $x \in U_{\mathcal{X}}$ if there exists a linear bounded map $D_x f : \mathcal{X} \to \mathcal{Y}$ such that

$$\lim_{h\to 0} \frac{||f(x+h) - f(x) - D_x f(h)||_{\mathcal{Y}}}{||h||_{\mathcal{X}}} = 0.$$

f is said to be C^1 if it is differentiable at every point $x \in U_{\mathcal{X}}$ and the map $Df : U_{\mathcal{X}} \to L(\mathcal{X}, \mathcal{Y})$ defined by $Df(x) = D_x f$ is continuous. Here $L(\mathcal{X}, \mathcal{Y})$ is the Banach space of linear bounded maps between $\mathcal{X}$ and $\mathcal{Y}$ with the operator norm.

A key tool in analysis is the *implicit function theorem.*

Theorem B.1 (Implicit function theorem (e.g. [37])) *Let $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$ be Banach spaces and $U_{\mathcal{X}}$, $U_{\mathcal{Y}}$, $U_{\mathcal{Z}}$ respective open sets with $0 \in U_{\mathcal{Z}}$. Let $f : U_{\mathcal{X}} \times U_{\mathcal{Y}} \to U_{\mathcal{Z}}$ be C^1 with Fréchet-derivative $D_{(x,y)}f$.*

Let $x_0 \in U_{\mathcal{X}}$, $y_0 \in \mathcal{Y}$ satisfy $f(x_0, y_0) = 0$ and assume that the linear map

$$\begin{aligned} D_y f|_{(x_0,y_0)} : \mathcal{Y} &\to \mathcal{Z}, \\ \hat{y} &\to D_{(x_0,y_0)} f(0, \hat{y}) \end{aligned}$$

is invertible, bounded and with bounded inverse. Then there exist open neighbourhoods $x_0 \in U_{x_0} \subset U_{\mathcal{X}}$ and $y_0 \in U_{y_0} \subset U_{\mathcal{Y}}$ and a C^1 map $g : U_{x_0} \to U_{y_0}$ such that $f(x, g(x)) = 0$ and, moreover, $f(x, y) = 0$ with $(x, y) \in U_{x_0} \times U_{y_0}$ implies $y = g(x)$.

In the context of partial differential equations, one important class of Banach spaces are the Hölder spaces.

Let $\Omega \subset \mathbb{R}^n$ be a domain and $f : \overline{\Omega} \to \mathbb{R}$. Let $\beta = (\beta_1, \cdots, \beta_n)$ be multi-index (i.e. $\beta_i \in \mathbb{N} \cup \{0\}$ for all $i \in \{1, \cdots n\}$) and define $|\beta| = \sum_{i=1}^n \beta_i$. Denote by $D^\beta f$ the partial derivative $D^\beta f = \partial_{x_1^{\beta_1}} \cdots \partial_{x_n^{\beta_n}} f$ when this exists. For $k \in \mathbb{N} \cup \{0\}$ we denote by $C^k(\overline{\Omega})$ the set of functions f with continuous derivatives $D^\beta f$ for all β with $|\beta| \leq k$.

Let $0 < \alpha \leq 1$. The function f is *Hölder continuous with exponent* α if

$$[f]_\alpha \equiv \sup_{\substack{x,y \in \overline{\Omega} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}$$

is finite. When $\alpha = 1$, the function is called Lipschitz continuous.

Definition B.2 *For $0 < \alpha \leq 1$ and $k \in \mathbb{N} \cup \{0\}$ the Hölder space $C^{k,\alpha}(\overline{\Omega})$ is the Banach space of all functions $u \in C^k(\overline{\Omega})$ for which the norm*

$$[f]_{k,\alpha} = \sum_{|\beta|=0}^{k} \sup_{\overline{\Omega}} |D^\beta f| + \max_{|\beta|=k} [D^\beta f]_\alpha$$

is finite.

The definition extends to Riemannian manifolds if we replace $|x - y|$ by the distance function $d(x, y)$ between two points.

The following result appearing in [58] (pages 448-449 and problem 17.2) is useful when we apply the implicit function theorem in Chapter 6.

Theorem B.3 *Let $\psi \in C^{2,\alpha}(\overline{\Omega})$ with $\Omega \subset \mathbb{R}$ a domain and consider the maps*

$$F : C^{2,\alpha}(\overline{\Omega}) \longrightarrow C^{0,\alpha}(\overline{\Omega})$$

and

$$\mathcal{F} : \Gamma = \overline{\Omega_2} \times \overline{\Omega} \longrightarrow \mathbb{R},$$

where $\Omega_2 \subset \mathbb{R}^3$ is a domain, which are related by

$$F(\psi)(x) = \mathcal{F}(\ddot{\psi}(x), \dot{\psi}(x), \psi(x), x).$$

Assume that $\mathcal{F} \in C^{2,\alpha}(\Gamma)$. Then F has continuous Fréchet derivative given by

$$\begin{aligned} D_\psi F(\varphi) &= \partial_1 \mathcal{F}\big|_{(\ddot{\psi}(x),\dot{\psi}(x),\psi(x),x)} \ddot{\varphi}(x) + \partial_2 \mathcal{F}\big|_{(\ddot{\psi}(x),\dot{\psi}(x),\psi(x),x)} \dot{\varphi}(x) \\ &\quad + \partial_3 \mathcal{F}\big|_{(\ddot{\psi}(x),\dot{\psi}(x),\psi(x),x)} \varphi(x). \end{aligned}$$

Consider a manifold S with metric g and let ∇ be the corresponding covariant derivative. Let a^{ij} be a symmetric tensor field , b^i a vector field and c a scalar. Consider a linear second order differential operator L on the form

$$L\psi = -a^{ij}(x)\nabla_i\nabla_j\psi + b^i(x)\nabla_i\psi + c(x)\psi, \tag{B.1}$$

Definition B.4 *L is* **elliptic** *at a point $x \in S$ if the matrix $[a^{ij}](x)$ is positive definite.*

Assume that S is orientable and denote by $<,>_{L^2}$ the L^2 inner product of two functions $\psi, \phi : S \to \mathbb{R}$ defined by $< \psi, \phi >_{L^2} \equiv \int_S \psi\phi\boldsymbol{\eta}_S$, where $\boldsymbol{\eta}_S$ is the (metric) volume form on S. Given a second order linear differential operator, the formal adjoint $L^\dagger$ is the linear second order differential operator which satisfies

$$< \psi, L^\dagger\phi >_{L^2} = < \phi, L\psi >_{L^2} .$$

for all pairs of smooth functions with compact support. A linear operator L is *formally self-adjoint* with respect to the product L^2 if $L^\dagger = L$.

When acting on the Hölder space $C^{2,\alpha}(S)$ for $0 < \alpha < 1$, the linear second order operator L becomes a bounded linear operator $L : C^{2,\alpha}(S) \to C^{0,\alpha}(S)$. The formal adjoint is also a map $L^\dagger : C^{2,\alpha}(S) \to C^{0,\alpha}(S)$. An *eigenvalue* of L is a number $\mu \in \mathbb{C}$ for which there exist functions $u, v \in C^{2,\alpha}(S)$ such that $L[u] + iL[v] = \mu\,(u + iv\ )$. The complex function $u + iv$ is called an *eigenfunction*.

The following lemma concerns the existence and uniqueness of the *principal eigenvalue* (i.e. the eigenvalue with smallest real part) of L and $L^\dagger$. This result is an adaptation of a standard result of elliptic theory to the case of compact connected manifolds without boundary (see Appendix B of [3]).

Lemma B.5 *Let L be a linear second order elliptic operator on a compact manifold S. Then*

1. *There is a real eigenvalue ϱ, called the principal eigenvalue, such that for any other eigenvalue μ the inequality $Re(\mu) \geq \varrho$ holds. The corresponding eigenfunction ϕ, $L\phi = \varrho\phi$ is unique up to a multiplicative constant and can be chosen to be real and everywhere positive.*

2. *The formal adjoint $L^\dagger$ (with respect to the L^2 inner product) has the same principal eigenvalue ϱ as L.*

For formally self-adjoint operators, the principal eigenvalue ϱ satisfies

$$\varrho = \inf_{\substack{\psi \in C^{2,\alpha}(S^2) \\ \psi \neq 0}} \frac{< \psi, L\psi >_{L^2}}{< \psi, \psi >_{L^2}}, \tag{B.2}$$

where the quotient $\frac{<\psi, L\psi>_{L^2}}{<\psi,\psi>_{L^2}}$ is called the Rayleigh-Ritz ratio of the function ψ. This formula, which reflects the connection between the eigenvalue problems and the variational problems, is also useful to obtain upper bounds for ϱ.

An important tool in the analysis of the properties of the elliptic operator L is the maximum principle. The standard formulations of the maximum principle for elliptic operators requires that the coefficient c in (B.1) is non-negative (see e.g. Section 3 of [58]). The following formulation of the maximum principle, which is more suitable for our purposes, requires non-negativity of the principal eigenvalue. Its proof can be found in Section 4 of [3].

Lemma B.6 *Consider a linear second order elliptic operator L on a compact manifold S with principal eigenvalue $\varrho \geq 0$ and principal eigenfunction ϕ and let ψ be a smooth function satisfying $L\psi \geq 0$ ($L\psi \leq 0$).*

1. *If $\varrho = 0$, then $L\psi \equiv 0$ and $\psi = C\phi$ for some constant C.*

2. *If $\varrho > 0$ and $L\psi \not\equiv 0$, then $\psi > 0$ ($\psi < 0$) all over S.*

3. *If $\varrho > 0$ and $L\psi \equiv 0$, then $\psi \equiv 0$.*

For surfaces S embedded in an initial data set (Σ, g, K), the outer null expansion θ^+ (also the inner null expansion θ^-) is a *quasilinear* second order elliptic operator[1] acting on the embedding functions of S. In this case, there also exists a maximum principle which is useful (see e.g. [4]).

Proposition B.7 *Let (Σ, g, K) be an initial data set and let S_1 and S_2 be two connected C^2-surfaces touching at one point $\mathfrak{p}$, such that the outer normals of S_1 and S_2 agree at $\mathfrak{p}$. Assume furthermore that S_2 lies to the outside of S_1, that is in direction of its outer normal near $\mathfrak{p}$, and that*

$$\sup_{S_1} \theta^+[S_1] \leq \inf_{S_2} \theta^+[S_2].$$

Then $S_1 = S_2$.

[1] A quasilinear second order elliptic operator Q has the form $Q\psi = -a^{ij}(x, \psi, \nabla\psi)\nabla_i\nabla_j\psi + b(x, \psi, \nabla\psi)$, with the matrix $[a^{ij}]$ being positive definite.

In particular, if two MOTS touch at one point and the outer normals agree there then the two surfaces must coincide. This maximum principle can be viewed as an extension of the maximum principle for minimal surfaces which asserts precisely that two minimal surfaces touching at one point are the same surface (see e.g. [51]).

We discuss next the Sard Lemma, which is needed at several places in the main text. First we define regular and critical value for a smooth map.

Let $f : \mathcal{N} \to \mathcal{M}$ be a smooth map. A point $\mathfrak{p} \in \mathcal{N}$ is a **regular point** if $D_{\mathfrak{p}} f : T_{\mathfrak{p}}\mathcal{N} \to T_{f(\mathfrak{p})}\mathcal{M}$ has maximum rank (i.e. $\mathrm{rank}(D_{\mathfrak{p}} f) = \min(n, m)$, where n is the dimension of $\mathcal{N}$ and m is the dimension of $\mathcal{M}$). A **critical point** $\mathfrak{p} \in \mathcal{M}$ is a point which is not regular. A point $\mathfrak{q} \in \mathcal{M}$ is a **regular value** if $f^{-1}(\mathfrak{q})$ is either empty or all $\mathfrak{p} \in f^{-1}(\mathfrak{q})$ are regular points. A point $\mathfrak{q} \in \mathcal{M}$ is a **critical value** if it is not a regular value.

We quote Theorem 1.2.2 in [93]

Theorem B.8 (Sard) *Let $\mathcal{N}$ and $\mathcal{M}$ be paracompact manifolds, then the set of critical values of a smooth map $f : \mathcal{N} \to \mathcal{M}$ has measure zero in $\mathcal{M}$.*

This theorem is equivalent to saying that the set of regular values of $f : \mathcal{N} \to \mathcal{M}$ is dense in $\mathcal{M}$.

For maps $f : \mathcal{N} \to \mathbb{R}$ the definition above states that $\mathfrak{p} \in \mathcal{N}$ is a critical point if and only if $df|_{\mathfrak{p}} = 0$. Let $\mathfrak{p} \in \mathcal{N}$ be a critical point and $H_{\mathfrak{p}}$ the Hessian at $\mathfrak{p}$ (i.e. $H_{\mathfrak{p}}(\vec{X}, \vec{Y}) = \vec{X}(\vec{Y}(f))|_{\mathfrak{p}}$). For any isolated critical point $\mathfrak{p} \in \mathcal{N}$ with non-degenerate Hessian, the Morse Lemma (see e.g. Theorem 7.16 in [48]) asserts that there exists neighbourhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ and coordinates $\{x_1, \cdots, x_n\}$ on $U_{\mathfrak{p}}$ such that $\mathfrak{p} = (0, \cdots 0)$ and f takes the form $f(x) = f(\mathfrak{p}) - (x_1)^2 - \cdots - (x_q)^2 + (x_{q+1})^2 + \cdots (x_n)^2$ where the signature of $H_{\mathfrak{p}}$ is $n - q$. For arbitrary critical points this Lemma has been generalized by Gromoll and Meyer [62]. The generalization allows for Hilbert manifolds of infinite dimensions. In the finite dimensional case Lemma 1 in [62] can be rewritten in the following form.

Lemma B.9 (Gromoll-Meyer splitting Lemma, 1969) *Let $\mathcal{N}$ be a manifold of dimension n and $f : \mathcal{N} \to \mathbb{R}$ a smooth map. Let $\mathfrak{p}$ be a critical point (not necessarily isolated) and $H_{\mathfrak{p}}$ the Hessian of f at $\mathfrak{p}$. Assume that the signature of $H_{\mathfrak{p}}$ is $\{\underbrace{+, \cdots, +}_{q}, \underbrace{-, \cdots, -}_{r}, \underbrace{0, \cdots, 0}_{n-q-r}\}$*

Then, there exists an open neighbourhood $U_{\mathfrak{p}}$ of $\mathfrak{p}$ and coordinates $\{x_1, \cdots, x_n\}$ such that $\mathfrak{p} = \{0, \cdots 0\}$ and f takes the form

$$f(x) = f(\mathfrak{p}) + (x_1)^2 + \cdots + (x_q)^2 - (x_{q+1})^2 - \cdots (x_{q+r})^2 + h(x_{q+r+1}, \cdots, x_n)$$

where h is smooth and this function, its gradient and its Hessian vanishes at $(x_{q+r+1} = 0, \cdots, x_n = 0)$.

Finally, the following result by Glaeser [60] is needed in Chapter 4 (proof of Proposition 4.3.14) when dealing with positive square roots of non-negative functions.

Theorem B.10 (Glaeser, 1963 [60]) *Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ be C^2 and satisfy $f \geq 0$ everywhere. If the Hessian of f vanishes everywhere on the set $F = \{\mathfrak{p} \in U, \text{such that } f(\mathfrak{p}) = 0\}$, then $g = +\sqrt{f}$ is C^1 on U.*

Bibliography

[1] L. Andersson, M. Mars, J. Metzger, W. Simon, "The time evolution of marginally trapped surfaces", *Class. Quantum Grav.* **26**, 085018(14pp) (2009). 3.4.2, 5.4

[2] L. Andersson, M. Mars, W. Simon, "Local existence of dynamical and trapping horizons", *Phys. Rev. Lett.* **95**, 111102(4pp) (2005). 2.2.2, 2.2.2

[3] L. Andersson, M. Mars, W. Simon, "Stability of marginally outer trapped surfaces and existence of marginally outer trapped tubes", *Adv. Theor. Math. Phys.* **12**, 853-888 (2008). 1, 2.2.2, 2.2.2, 2.2.2, 3.1, 4.3.3, B, B

[4] L. Andersson, J. Metzger, "The area of horizons and the trapped region", *Commun. Math. Phys.* **290**, 941-972 (2009). 1, 2.2.3, 2.2.31, 3.5, 4.4, B

[5] R. Arnowitt, S. Deser, C.W. Misner, "The dynamics of general relativity", in *Gravitation: An introduction to current research*, Ed. L. Witten, pp. 227-265, Wiley (1962). 2.3

[6] A. Ashtekar, G.J. Galloway, "Some uniqueness results for dynamical horizon", *Adv. Theor. Math. Phys.* **9**, 1-30 (2005). 3.1

[7] T. Bäckdahl, J.A. Valiente-Kroom, "On the construction of a geometric invariant measuring the deviation from Kerr data", *Annales Henri Poincaré* **11**, 1225-1271 (2010). 1

[8] T. Bäckdahl, J.A. Valiente-Kroom, "The "non-Kerrness" of domains of outer communication of black holes and exteriors of star", *Proc. Roy. Soc. Lond. A* **467**, 1701-1718 (2011). 1

[9] R. Bartnik, "The mass of an asymptotically flat manifold", *Commun. Pure and App. Math* **39**, 661-693 (1986). 2.3

[10] R. Bartnik, P.T. Chruściel, "Boundary value problems for Dirac-type equations, with applications", *Journal für die reine und angewandte Mathematik (Crelle's Journal)* **579**, 13-73 (2005). 2.3, 2.4.12

[11] R. Bartnik, J. McKinnon, "Particle solutions of the Einstein-Yang-Mills equations", *Phys. Rev. Lett.* **61**, 141-144 (1988). 2.4

[12] J.K. Beem, P.E. Ehrlich, S. Markvorsen, "Timelike isometries and Killing fields", *Geom. Dedicata* **26**, 247-258 (1988). 3.4

[13] R. Beig, P.T. Chruściel, "Killing initial data", *Class. Quantum Grav.* **14**, A83-A92 (1997). 1, 4.2.1, 4.2.1, 4.2.1, 4.2.1

[14] I. Ben-Dov, "Outer trapped surfaces in Vaidya spacetimes", *Phys. Rev. D* **75**, 064007(33pp) (2007). 2.2.1, 3.1

[15] I. Bengtsson, J.M.M. Senovilla, "A Note on trapped Surfaces in the Vaidya Solution", *Phys. Rev. D* **79**, 024027(6pp) (2009). 2.2.1, 3.1

[16] I. Bengtsson, J.M.M. Senovilla, "The region with trapped surfaces in spherical symmetry, its core, and their boundaries", *Phys. Rev. D* **83**, 044012(30pp) (2011). 3.1

[17] R.H. Boyer, "Geodesic Killing orbits and bifurcate Killing horizons", *Proc. Roy. Soc. A* **311**, 245-253 (1969). 1, 2.4.9, 4.3.3

[18] S. Bravo, *Métodos matemáticos avanzados para científicos e ingenieros* (Manual 48 UEx), Servicio de Publicaciones de la UEx (2006). 6.2.1

[19] H.L. Bray "Proof of the Riemannian Penrose inequality using the positive mass theorem", *J. Diff. Geom.* **59**, 177-267 (2001). 2.3

[20] H.L. Bray, M. Khuri, "P.D.E.'s which imply the Penrose inequality", *arXiv:0905.2622* (2009). 1, 1, 2.2.1, 2.3, 2.3, 6.1, 6.1

[21] H.L. Bray, M. Khuri, "A Jang equation approach to the Penrose inequality", *Discrete and Continuous Dynamical Systems A* **27**, 741-766 (2010). 6.2.2

[22] P. Breitenlohner, D. Maison, G. Gibbons, "4-dimensional black holes from Kaluza-Klein", *Commun. Math. Phys.* **120**, 295-333 (1988). 2.4

[23] G. Bunting, A.K.M. Masood-ul-Alam, "Nonexistence of multiple black holes in asymptotically euclidean static vacuum space-time", *Gen. Rel. Grav.* **19**, 147-154 (1987). 1, 2.4, 4.1, 5.4

[24] A. Carrasco, M. Mars, "On marginally outer trapped surfaces in stationary and static spacetimes", *Class. Quantum Grav.* **25**, 055011(19pp) (2008). 1, 3.1, 4.1

[25] A. Carrasco, M. Mars, "Marginally outer trapped surfaces in stationary initial data", in *Proceedings of the Spanish Relativity Meeting - ERE 2007*, EAS Publication Series **30**, 261-264 (2008). 1, 3.1, 4.1

[26] A. Carrasco, M. Mars, "Stability in marginally outer trapped surfaces in spacetimes with symmetries" *Class. Quantum Grav.* **26**, 175002(19pp) (2009). 1, 3.1, 3.5

[27] A. Carrasco, M. Mars, "Marginally outer trapped surfaces and symmetries", in *Proceedings of the Spanish Relativity Meeting - ERE2008, Physics and mathemathics of gravitation*, AIP Conference Proceedings **112**, 221-224 (2009). 1, 3.1

[28] A. Carrasco, M. Mars, "A counterexample to a recent version of the Penrose conjecture ", *Class. Quantum Grav.* **27**, 062001(10pp) (2010). 1, 6.1

[29] A. Carrasco, M. Mars, "On a recent proposal of the Penrose inequality", in *Proceedings of the Spanish Relativity Meeting- ERE2009*, J. Phys. Conference series **229**, 012027(4pp) (2010). 1, 6.1

[30] A. Carrasco, M. Mars, "Uniqueness of static spacetimes containing marginally outer trapped surfaces", *in preparation.* 5.1

[31] A. Carrasco, M. Mars, "Uniqueness of static spacetimes containing MOTS", in *Proceedings of the Spanish Relativity Meeting- ERE2010*, to be printed. 5.1

[32] B. Carter, "Killing horizons and orthogonally transitive groups in spacetime", *J. Math. Phys.* **10**, 70-81 (1969). 2.4.8

[33] B. Carter, "Axisymmetric black holes has only two degrees of freedom", *Phys. Rev. Lett.* **26**, 331-332 (1971). 2.4

[34] B. Carter, "Black holes equilibrium states", in *Black Holes*, Ed. C. DeWitt and B.S. DeWitt, Gordon & Breach (1973). 2.4

[35] S. Chandrasekhar, "The density of white dwarf stars", *Phil. Mag.* **11**, 592-596 (1931). 1

S. Chandrasekhar, "The maximun mass of ideal white dwarf stars", *Astrophys. J.* **74**, 81-82 (1931).

L.D. Landau, "On the theory of stars", *Phys. Z. Sowjetunion* **1**, 285-288 (1932).

J.R. Oppenheimer, G. Volkoff, "On massive neutron cores", *Phys. Rev.* **56**, 374-381 (1939).

[36] I. Chavel, *Riemannian geometry, a modern introduction* (Cambridge Studies in Advanced Mathematics **98**), Cambridge University Press (2006). 2.2.1, 2.2.2, 6.2.2

[37] Y. Choquet-Bruhat, C. Dewitt-Morette, M. Dillard-Bleick, *Analysis, manifolds and physics*, North-Holland (1977). B.1

[38] D. Christodoulou, "The instability of naked singularities in the gravitational collapse of a scalar field", *Ann. Math.* **149**, 183-217 (1999). 1

[39] P.T. Chruściel, "The classification of static vacuum space-times containing an asymptotically flat spacelike hypersurface with compact interior", *Class. Quantum Grav.* **16**, 661-687 (1999). 1, 2.4, 2.4.1, 2.4.9, 2.4.11, 2.4.1

[40] P.T. Chruściel, "Towards a classification of static electro-vacuum space-times containing an asymptotically flat spacelike hypersurface with compact interior", *Class. Quantum Grav.* **16**, 689-704 (1999). 1, 2.4, 2.4.1

[41] P.T. Chruściel, "The classification of static vacuum space-times containing an asymptotically flat spacelike hypersurface with compact interior", *arXiv:gr-qc/9809088v2 (corrigendum to arXiv:gr-qc/9809088 submitted in 2010).* 1, 2.4.1, 2.4.1, 2.4.1

[42] P. Chruściel, J. Lopes Costa, "On uniqueness of stationary vacuum black holes" in *Géométrie Différentielle, Physique Mathématique, Mathématique et Société, Volume en l' honneur de Jean Pierre Bourguignon*, Ed. O. Hijazi, *Astérisque* **321**, pp. 195-265 (2008). 2.3, 2.4, 2.4.1

[43] P.T. Chruściel, E. Delay, G.J. Galloway, R. Howard, "Regularity of horizons and the area theorem", *Annales Henri Poincaré* **2**, 1779-1817 (2001). 2.3

[44] P.T. Chruściel, G.J. Galloway, "Uniqueness of static black holes without analyticity", *Class. Quantum Grav.* **27** 152001(6pp) (2010). 2.4.1, 2.4.15

[45] P.T. Chruściel, P. Tod, "The classification of static electro-vacuum space-times containing an asymptotically flat spacelike hypersurface with a compact interior", *Commun. Math. Phys.* **271**, 577-589 (2007). 1, 2.4, 2.4.2, 2.4.1

[46] B. Coll, "On the evolution equations for Killing fields", *J. Math. Phys.* **18**, 1918-1922 (1997). 1, 4.2.1, 5.2

[47] B. Coll, S. Hildebrant, J.M.M. Senovilla, "Kerr-Schild symmetries", *Gen. Rel. Grav.* **33**, 649-670 (2001). 3.1

[48] L. Conlon, *Differentiable manifolds*, Birkhäuser (2008). B

[49] J. Lopes Costa, "On black hole uniqueness theorems", *Ph. D. Thesis* (2010). 2.3, 2.4

[50] M. Dafermos, "Spherically symmetric spacetimes with a trapped surface", *Class. Quantum Grav.* **22**, 2221-2232 (2005). 1

[51] U. Dierkes, S. Hildebrandt, A.J. Tromba, *Regularity of minimal surfaces*, Springer Verlag (2010). B

[52] D.M. Eardley, "Black hole boundary conditions and coordinate conditions", *Phys. Rev. D* **57**, 2299(6pp) (1998). 2.2.1

[53] M. Eichmair "Existence, regularity and properties of generalized apparent horizons", *Commun. Math. Phys.* **294**, 745-760 (2009). 1, 2.2.3, 2.2.3, 2.2.32, 2.3

[54] A. Einstein, "Zur allgemeinen Relativitästherorie", *Preuss. Akad. Wiss. Berlin, Sizber.*, 778-786 (1915). 1

A. Einstein, "Der Feldgleichungen der Gravitation", *Preuss. Akad. Wiss. Berlin, Sizber.*, 844-847 (1915).

[55] L.P. Eisenhart, *Riemannian geometry*, Princeton University Press (1966). 4.2.2

[56] M.E. Feighn, "Separation properties of codimension-1 inmersions", *Topology* **27**, 319-321 (1988). 5.3, 5.3

[57] G.W. Gibbons, S.W. Hawking, G.T. Horowitz, M.J. Perry, "Positive mass theorems for black holes", *Commun. Math. Phys.* **88**, 295-308, (1983). 2.3

[58] D. Gilbarg, N.S. Trudinger, *Elliptic partial differential equations of second order*, Springer Verlag (1983). 4.3.3, B, B

[59] E. Giusti, *Minimal surfaces and functions of bounded variation*, Birkhäuser (1984). 2.2.3

[60] G. Glaeser, "Racine carrée d'une fonction différentiable", *Ann. Inst. Fourier Grenoble* **13**, 203-210 (1963). B, B.10

[61] G.A. González, R. Vera, "Conformastat electrovacuum spacetimes", *J. Phys.: Conf. Ser.* **229**, 012040(4pp) (2010). 2.4.1

G.A. González, R. Vera, "A local characterisation for static charged black holes", *Class. Quantum Grav.* **28**, 025008(28pp) (2011).

[62] D. Gromoll, W. Meyer, "On differentiable functions with isolated critical points", *Topology* **8**, 361-369 (1969). B

[63] S.W. Hawking, "Gravitational radiation from colliding black holes", *Phys. Rev. Lett.* **26**, 1344-1346, (1971). 2.3

[64] S.W. Hawking, "Black holes in general relativity", *Commun. Math. Phys.* **25**, 152-166 (1972). 2.3

[65] S.W. Hawking, G.F.R. Ellis, *The large scale structure of space-time* (Cambridge monographs on mathematical physics), Cambridge University Press (1973). 1, 2.2.1, 2.4, 3.4.2

[66] M. Heusler, "On the uniqueness of the Reissner-Nordström solution with electric and magnetic charge", *Class. Quantum Grav.* **11**, L49-53 (1994). 2.4.1

[67] M. Heusler, *Black hole uniqueness theorems* (Cambridge Lecture Notes in Physics **6**), Cambridge University Press (2006). 1, 2.4.1, 4.3.3

[68] M.W. Hirsch, *Differential Topology* (Graduate Texts in Mathematics **33**), Springer Verlag (1997). A

[69] G.T. Horowitz, "The positive mass theorem and its extensions" in *Asymptotic behaviour of mass and spacetime geometry*, Ed. F.J. Flaherty (Springer Lecture Notes in Physics **202**), pp. 1-20, Springer Verlag (1984). 2.3

[70] G. Huisken, T. Ilmanen, "The inverse mean curvature flow and the Riemannian Penrose inequality", *J. Diff. Geom.* **59**, 353-437 (2001). 2.3, 5.4

[71] W. Israel, "Event horizons in static vacuum space-times", *Phys. Rev* **164**, 397-399 (1967). 1, 2.4

[72] W. Israel, "Event horizons in static electro-vac space-times", *Commun. Math. Phys.* **8**, 245-260 (1968). 1, 2.4

[73] W. Israel, "Differential forms in general relativity", *Commun. Dublin Inst. Adv. Stud. A* **19** (1970). 4.2.3

[74] P.S. Jang, "On the positivity of energy in general relativity", *J. Math. Phys.* **19**, 1152-1155 (1978). 2.3

[75] K. Karkowski, E. Malec, "The general Penrose inequality: lessons from numerical evidence", *Acta Physica Polonica* **B36**, 59-73 (2005). 2.3

[76] M. Khuri, "A note on the nonexistence of generalized apparent horizons in Minkowski space", *Class. Quantum Grav.* **26**, 078001(5pp) (2009). 2.3

[77] M. Kriele, S.A. Hayward, "Outer trapped surfaces and their apparent horizon", *J. Math. Phys.* **38**, 1593-1604 (1997). 1, 3.1, 3.5, 3.5.1, 3.5

[78] J.M. Lee, *Manifolds and differential geometry* (Graduate Studies in Mathematics **107**), American Mathematical Society (2009). 2.4.1

[79] M. Mars, "A spacetime characterization of the Kerr metric", *Class. Quantum Grav.* **16**, 2507-2523 (1999). 4.3.2

[80] M. Mars, "Present status of the Penrose inequality", *Class. Quantum Grav.* **26**, 193001(59pp) (2009). 1, 2.3

[81] M. Mars, "Uniqueness properties of the Kerr metric", *Class. Quantum Grav.* **17**, 3353-3374 (2000). 1

[82] M. Mars, J.M.M. Senovilla, "Trapped surfaces and symmetries", *Class. Quantum Grav.* **20**, L293-L300 (2003). 1, 3.1, 3.4, 3.4, 3.4, 3.4.2

[83] M. Mars, W. Simon, "On uniqueness of static Einstein-Maxwell-Dilaton black holes", *Adv. Theor. Math. Phys.* **6**, 279-306 (2002). 1, 2.4

[84] A.K.M. Masood-ul-Alam, "Uniqueness proof of static black holes revisited", *Class. Quantum Grav.* **9**, L53-L55 (1992). 1, 2.4, 2.4.1

[85] A.K.M. Masood-ul-Alam, "Uniqueness of a static charged dilaton black hole", *Class. Quantum Grav.* **10**, 2649-2656 (1993). 1, 2.4

[86] P.O. Mazur, "Proof of uniqueness of the Kerr-Newman black hole solution", *J. Phys. A* **15**, 3173-3180 (1982). 2.4

[87] P. Miao, "Positive mass theorem on manifolds admiting corners along a hypersurface", *Class. Quantum Grav.* **16**, 1163-1182 (2002). 2.3

[88] P. Miao, "A remark on boundary effects in static vacuum initial data sets", *Class. Quantum Grav.* **22**, L53-L59 (2005). 1, 4.1

[89] H. Müller zum Hagen, D.C. Robinson, H.J. Seifert, "Black holes in static vacumm space-times", *Gen. Rel. Grav.* **4**, 53-78 (1973). 1, 2.4

[90] H. Müller zum Hagen, D.C. Robinson, H.J. Seifert, "Black holes in static electrovac space-times", *Gen. Rel. Grav.* **5**, 61-72 (1974). 1, 2.4

[91] C. Camacho, A. Lins Neto, *Geometric theory of foliations*, Birkhäuser (1984). 4.4

[92] R.P.A.C. Newman, "Topology and stability of marginal 2-surfaces", *Class. Quantum Grav.* **4**, 277-290 (1987). 2.2.2

[93] L. Nirenberg, *Topics in Nonlinear Functional Analysis*, Notes by R.A. Artino (Courant Lecture Notes **6**), American Mathematical Society (2000). B

[94] B. O'Neill, *Semi-Riemannian geometry with applications to relativity*, Elsevier/Academic Press (1983). 2.2.1, 2.2.1

[95] R. Penrose, "Gravitational collapse and space-time singularities", *Phys. Rev. Lett.* **10**, 66-68 (1965). 1

S.W. Hawking, R. Penrose, "The singularities of gravitational collapse and cosmology", *Proc. Roy. Soc. Lond.* **A314**, 529-548 (1970).

[96] R. Penrose, "Gravitational collapse – the role of general relativity", *Nuovo Cimiento* **1**, 252-276 (1965). 1

[97] R. Penrose, “Naked singularities”, *Ann. N. Y. Acad. Sci.* **224**, 125-134 (1973). 1, 2.3, 2.3

[98] I. Rácz, R.M. Wald, “Extensions of spacetimes with Killing horizons”, *Class. Quantum Grav.* **9**, 2643-2656 (1992). 1, 2.4.5, 5.2

[99] D.C. Robinson, “Uniqueness of the Kerr black hole”, *Phys. Rev. Lett.* **34**, 905-906 (1975). 2.4

[100] D.C. Robinson, “A simple proof of the generalization of the Israel’s theorem”, *Gen. Rel. Grav.* **8**, 695-698 (1977). 1, 2.4

[101] J.J. Rotman, *An introduction to algebraic topology*, Springer Verlag (1991). 5.3

[102] P. Ruback, “A new uniqueness theorem for charged black holes”, *Class. Quantum Grav.* **5**, L155-L159 (1988). 1, 2.4, 2.4.1

[103] R. Ruffini, J.A. Wheeler, “Introducing the Black Hole”, *Physics Today* **24** 30-41 (1971). 1

[104] R. Schoen, S.Y. Yau, “Proof of the positive mass theorem II”, *Commun, Math. Phys.* **79**, 231-260 (1981). 2.3, 2.3

[105] J.M.M. Senovilla, “Singularity theorems and their consequences”, *Gen. Rel. Grav.* **29**, 701-846 (1997). 1, 3.4.1

[106] J.M.M. Senovilla, “Classification of spacelike surfaces in spacetime”, *Class. Quantum Grav.* **24**, 3091-3124 (2003). 2.2.1

[107] J.M.M. Senovilla, “On the existence of horizons in spacetimes with vanishing curvature invariants”, *J. High Energy Physics* **11**, 046(7pp) (2003). 1, 3.1

[108] W. Simon, “A simple proof of the generalized Israel theorem”, *Gen. Rel. Grav.* **17**, 761-768 (1985). 1, 2.4

[109] W. Simon, “Radiative Einstein-Maxwell spacetimes and ’no-hair’ theorems”, *Class. Quantum Grav.* **9**, 241-256 (1992). 1, 2.4

[110] N. Straumann, *General relativity: With applications to astrophysics*, Springer Verlag (2004). 2.3

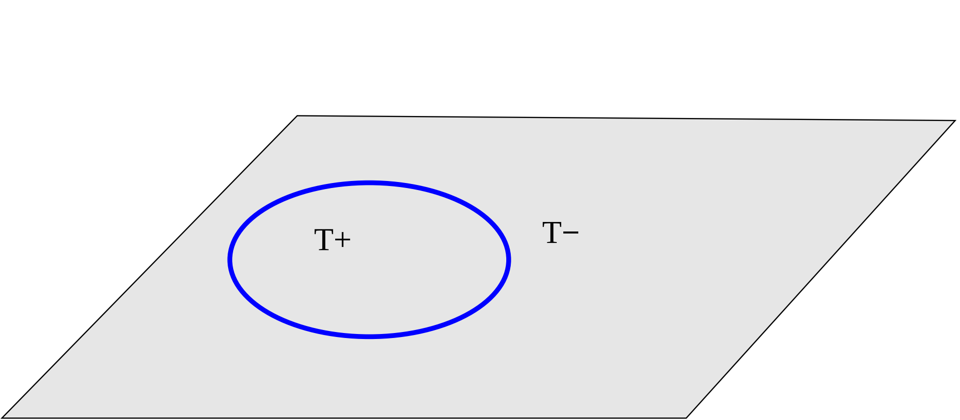
[111] C.V. Vishveshwara, "Generalization of the 'Schwarzschild surface' to arbitrary static and stationary metrics", *J. Math. Phys.* **9**, 1319-1322 (1968). 2.4.8

[112] R.M. Wald, *General Relativity*, The University of Chicago Press (1984). 1, 2.1, 2.1, 2.2.1, 2.2.2, 2.3, 2.3, 2.4.1, 4.1, 4.3.2

[113] R.M. Wald, "Gravitational collapse and cosmic censorship", in *Black Holes, Gravitational Radiation and the Universe*, Ed. B.R. Iyer and B. Bhawal (Fundamental Theories of Physics **100**), pp. 69-85, Kluwer Academic (1999). 1

[114] E. Witten, "A new proof of the positive energy theorem", *Commun. Math. Phys.* **80**, 381-402 (1981). 2.3

[115] A.J. Zomorodian, *Topology for computing*, (Cambridge Monographs on applied and computational mathematics **16**), Cambrigde University Press (2005). 5.3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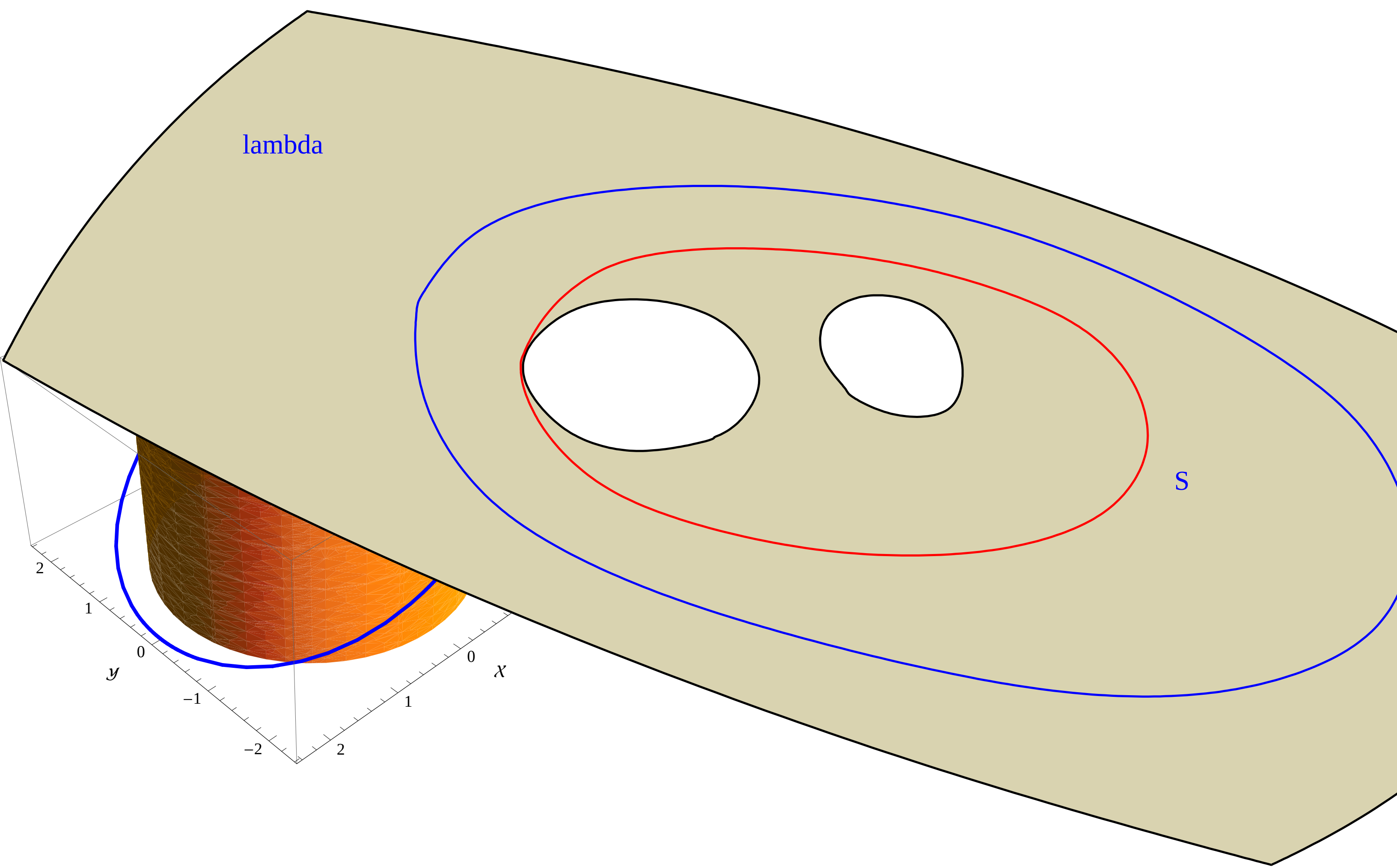

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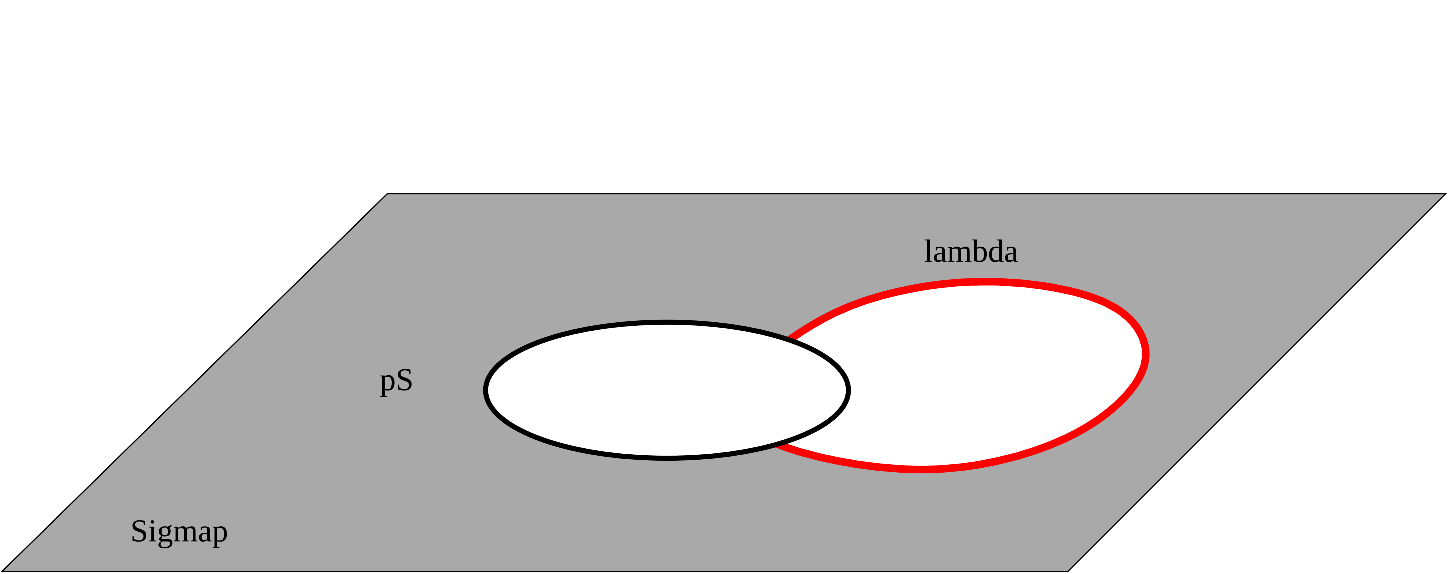
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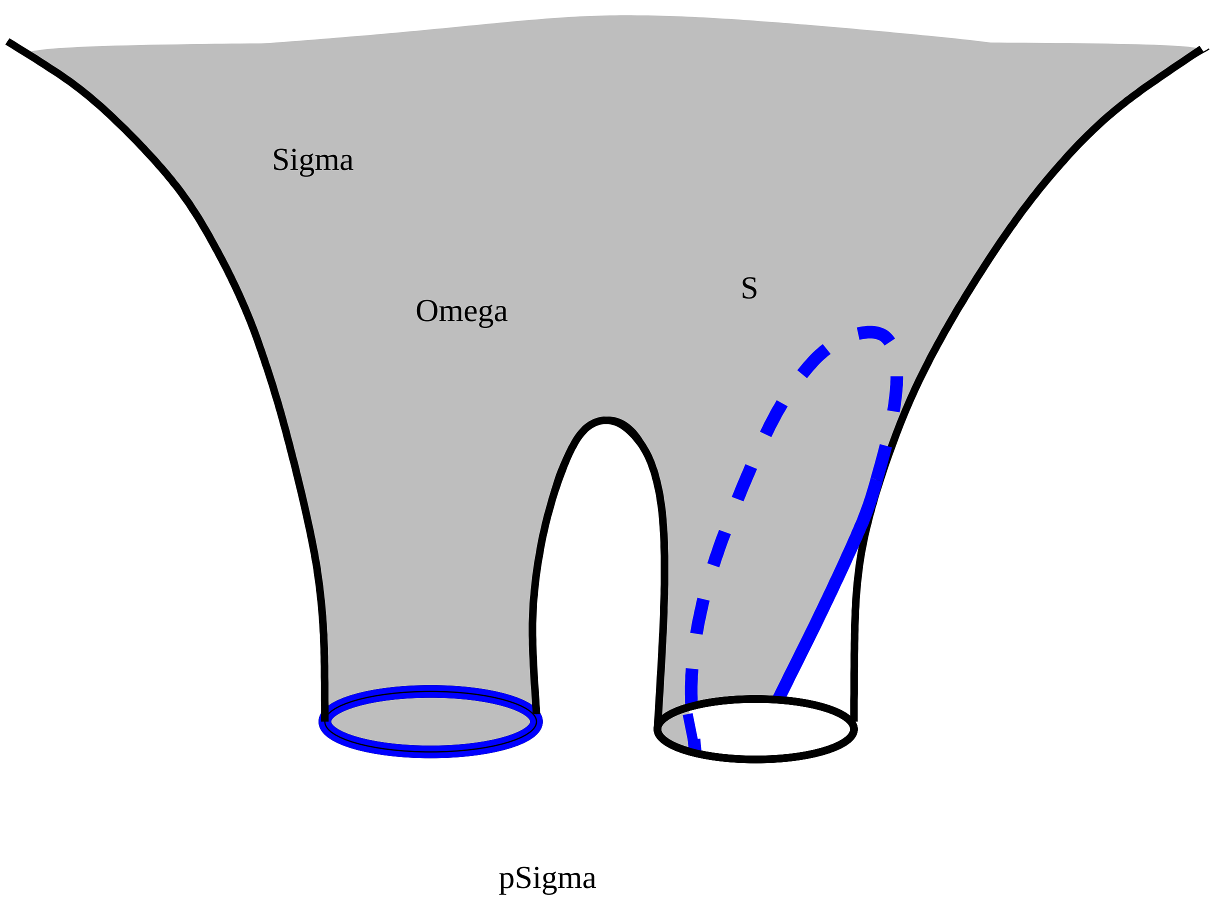

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